THE COHOMOLOGY OF DELIGNE-LUSZTIG VARIETIES FOR THE GENERAL LINEAR GROUP

(PRELIMINARY VERSION)

SASCHA ORLIK

Abstract. We propose two inductive approaches for determining the cohomology of Deligne-Lusztig varieties in the case of $G = \mathrm{GL}_n$.

1. Introduction

In 1976 Deligne and Lusztig [DL] introduced certain locally closed subvarieties in flag varieties over finite fields which are of particular importance in the representation theory of finite groups of Lie type. They proved that their Euler-Poincaré characteristic considered as a virtual representation of the corresponding finite group detect all irreducible representations. However, a description of the individual cohomology groups of Deligne-Lusztig varieties has been determined since then only in a few special cases cf. [L2, DMR, DM, Du], where in contrast the intersection cohomology groups of their Zariski closures are known [L3]. In this paper we propose two inductive approaches for determining all of them in the case of $G = \mathrm{GL}_n$ (resp. for reductive groups of Dynkin type A_{n-1}). Although the key ideas work for other (split) reductive groups as well, we have decided to treat here only the case of the general linear group since things are more concrete in this special situation.

For a split reductive group $\mathbf{G}$ defined over $k = \mathbb{F}_q$, let X be the set of all Borel subgroups of $\mathbf{G}$. Let $F : X \to X$ be the Frobenius map over $\mathbb{F}_q$. The Deligne-Lusztig variety associated to an element $w \in W$ of the Weyl group is the locally closed subset of X given by

$$X(w) = \{x \in X \mid \mathrm{inv}(x, F(x)) = w\}.$$

Here $\mathrm{inv} : X \times X \longrightarrow W$ is the relative position map. Then $X(w)$ is a smooth quasi-projective variety defined over $\mathbb{F}_q$. It is naturally equipped with an action of $G = \mathbf{G}(k)$ and has dimension equal to the length of w. The ℓ-adic cohomology with compact support $H^*_c(X(w)) := H^*_c(X(w), \overline{\mathbb{Q}}_\ell)$ has therefore the structure of a $G \times \mathrm{Gal}(\overline{k}/k)$-module.

Date: February 5, 2013.

Let $\mathbf{G} = \mathrm{GL}_n$. For the determination of the cohomology of DL-varieties, we make heavily use of certain maps $\gamma : X_1 \longrightarrow X(w')$ resp. $\delta : X_2 \longrightarrow X(sw')$ introduced in [DL, Theorem 1.8]. Here $w, w' \in W$ and s is a simple reflection with $w = sw's$ and $\ell(w) = \ell(w') + 2$. Further X_1 is a closed subset of $X(w)$ and X_2 denotes its open complement. It is proved in loc.cit that γ is a $\mathbb{A}^1$-bundle whereas δ is a $\mathbb{G}_m$-bundle. In this paper we consider instead of the map δ its look-alike $X_2 \longrightarrow X(w's)$. The above maps[1] extend to $\mathbb{P}^1$-bundles $X_2 \cup X(sw') \cup X(w's) \longrightarrow X(w's)$ and $X_1 \cup X(w') \longrightarrow X(w')$ which glue in turn to a $\mathbb{P}^1$-bundle

$$\gamma : X(Q) \longrightarrow X(w's) \cup X(w')$$

where $X(Q) = X(w) \cup X(sw') \cup X(w's) \cup X(w')$. Here $\gamma_{|X(w's)\cup X(w')} = \mathrm{id}$ whereas the restriction of γ to $Z := X(w) \cup X(sw')$ is a $\mathbb{A}^1$-bundle over the base $Z' := X(w's) \cup X(w')$. In particular, we deduce that

$$H^i_c(X(Q)) = H^i_c(Z') \oplus H^{i-2}_c(Z')(-1)$$

for all integers $i \geq 2$.

The quadruple $Q = \{w', sw', w's, w\} \subset W$ is a square in the sense of [BGG]. The notion of a square appears in the theory of BGG-resolutions of finite-dimensional Lie algebra representations. It seems to be also useful in the study of the cohomology of Deligne-Lusztig varieties. We consider more generally hypersquares in W and even in the monoid F^+ which is freely generated by the subset S of simple reflections in W. In fact we work more generally with DL-varieties and their Demazure compactifications attached to elements in F^+ in the spirit of [DMR]. More precisely, let $w = s_{i_1} \cdots s_{i_r}$ be a fixed reduced decomposition of $w \in W$ and let $\overline{X}(w)$ be the associated Demazure compactification of $X(w)$. This variety is equipped with a compatible action of G. We consider the closed complement of $X(w)$ in $\overline{X}(w)$ which is - as already observed in [DL] - a union of smooth equivariant divisors. We analyse the resulting spectral sequence converging to the cohomology of $X(w)$. The crucial point is that the intersection of these divisors is again a compactification of a DL-variety attached to some subexpression of $s_{i_1} \cdots s_{i_r} \in F^+$. Concretely the spectral sequence has the shape

$$E_1^{p,q} = \bigoplus_{v \preceq w, \ell(v) = \ell(w) - p} H^q(\overline{X}(v)) \Longrightarrow H^{p+q}_c(X(w)).$$

Another feature is that if $w = sw's \in F^+$, then $\overline{X}(w)$ is a $\mathbb{P}^1$-bundle over $\overline{X}(w's)$. This comes about from the fact that $\overline{X}(w)$ is paved by DL-varieties attached to squares of the special type as above. So by induction on the length of $w's$ we know the cohomology of the compactification $\overline{X}(w)$. Of course not every element w in F^+ has the shape $w = sw's$,

[1]which seems to be known to the experts

but by using a result of [GKP], every element can be transformed into such an element by applying the usual Weyl group relations and a cyclic shift operator. We study henceforth the effect on the cohomology by these operations. The start of induction is given by elements of minimal length in their conjugacy classes, i.e. by Coxeter elements in Levi subgroups of G. This is one reason why we deal only with reductive groups of Dynkin type A_{n-1}. In this case the Demazure compactification of the standard Coxeter element can be considered as one of the Drinfeld halfspace $\Omega^n = \mathbb{P}^{n-1} \setminus \bigcup_{H/\mathbb{F}_q} H$ (complement of all $\mathbb{F}_q$-rational hyperplanes in the projective space of lines in $V = \mathbb{F}^n$), cf. [Dr], and may be realised as a sequence of blow ups as it comes up in the arithmetic theory of Ω^n over a local field [Ge, GK, I].

Let us formulate the main theorems.

Theorem. *i) For all $w \in F^+$, we have $H^i(\overline{X}(w)) = 0$ for odd i.*

ii) Let $w = sw's$ with $s \in S$ and $w' \in F^+$. Then there are decompositions $H^i(\overline{X}(w)) = H^i(\overline{X}(w's)) \oplus H^{i-2}(\overline{X}(w's))(-1) = H^i(\overline{X}(sw')) \oplus H^{i-2}(\overline{X}(sw'))(-1)$.

iii) The action of the Frobenius on $H^i(\overline{X}(w))$ and $H^i_c(X(w))$ is semi-simple for all $w \in F^+$ and for all $i \geq 0$.

iv) For all $i \geq 0$, the cycle map $A^i(\overline{X}(w))_{\overline{\mathbb{Q}}_\ell} \longrightarrow H^{2i}(\overline{X}(w))$ is an isomorphism $\big($where $A^i(\overline{X}(w))$ is the Chow group of $\overline{X}(w)$ in degree $i\big)$.

An analogue of part ii) of this theorem is already formulated in [DMR, Prop. 3.2.3] in the case of Weyl group elements.

It turns out that the cohomology of the varieties $\overline{X}(w)$ is similar to the classical situation of Schubert varieties. Let $\prec$ be the Bruhat order on F^+. For a parabolic subgroup $P \subset G$, let $i_P^G = \mathrm{Ind}_P^G(\overline{\mathbb{Q}_\ell})$ be the induced representation of the trivial one. By part ii) of the above theorem and by studying the effect of the usual Weyl group relations on the Demazure compactifications of DL-varieties we are able to deduce the next statement.

Theorem. *Let $w \in F^+$. Then the cohomology of $\overline{X}(w)$ can be written as*

$$H^*(\overline{X}(w)) = \bigoplus_{z \preceq w} i^G_{P^w_z}(-\ell(z))[-2\ell(z)]$$

for certain std parabolic subgroups $P^w_z \subset G$.

The gradings are not canonical as there are in general plenty of choices. Moreover, if $v \prec w$, the restriction map $H^i(\overline{X}(w)) \longrightarrow H^i(\overline{X}(v))$ is highly not a graded homomorphism. But we shall see that it is at least quasi-isomorphic to such a homomorphism for an appropriate choice of gradings. More generally, we analyse in this respect the spectral sequence attached

to $\overline{X}(w)$ and its divisors. By weight reasons the above spectral sequence degenerates in E_1 and we believe that it can be evaluated via the following approach.

Conjecture. *Let $w \in F^+$ and fix an integer $i \geq 0$. For $v \preceq w$, there are gradings $H^{2i}(\overline{X}(v)) = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} i^G_{P^v_z}(-\ell(z))[-2\ell(z)]$ such that the complex*

$$E_1^{\bullet,2i} : H^{2i}(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-2}} H^{2i}(\overline{X}(v)) \longrightarrow \cdots \longrightarrow H^{2i}(\overline{X}(e))$$

is quasi-isomorphic to a direct sum $\bigoplus_{\substack{z \preceq w \\ \ell(z)=i}} H(\cdot)_z$ of complexes of the shape

$$H(\cdot)_z : i^G_{P^w_z} \to \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-1}} i^G_{P^v_z} \to \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-2}} i^G_{P^v_z} \to \cdots \to i^G_{P^e_z}$$

(Here the maps $i^G_{P^v_z} \longrightarrow i^G_{P^v_{z'}}$ in the complex are induced - up to sign - by the double cosets of 1 in $W_{P^v_z} \setminus W / W_{P^v_{z'}}$ via Frobenius reciprocity. Further $i^G_{P^v_z} = (0)$ if $z \npreceq v$.).

We are able to prove the conjecture is some cases. Additionally, we prove how to reduce the issue to the case where w is again of the form $w = sw's$.

Theorem. *i) The conjecture is true for Coxeter elements w.*

ii) If w is arbitrary, then the conjecture is true for $i \in \{0, 1, \ell(w)-1, \ell(w)\}$.

Of course the cases $i = 0$ and $i = \ell(w)$ are trivial. As a consequence we derive an inductive formula for the Tate twist -1-contribution $H^*_c(X(w))\langle -1\rangle$ of the cohomology of a DL-variety $X(w)$. For a parabolic subgroup P of G, let v^G_P be the corresponding generalized Steinberg representation.

Corollary. *Let $w = sw's \in F^+$ with $\mathrm{ht}(sw') \geq 1$. Then*

$$H^*_c(X(w))\langle -1\rangle = \begin{cases} H^*_c(X(sw'))\langle -1\rangle[-1] & \text{if } s \in \mathrm{supp}(w') \\ H^*_c(X(sw'))\langle -1\rangle[-1] \bigoplus v^G_{P(s)}(-1)[-\ell(w)] & \text{if } s \notin \mathrm{supp}(w') \end{cases}.$$

Here $\mathrm{supp}(w')$ denotes the set of simple reflections appearing in w' whereas $P(s)$ is the parabolic subgroup of G generated by B and s. Moreover, ht is the height function on F^+. It has the property that $\mathrm{ht}(w) = \mathrm{ht}(w') + 1$ if $w = sw's$ as above. Further $\mathrm{ht}(w) = 0$ if w is minimal length in its conjugacy class. The start of the inductive formula is hence given by height one elements. Here we are even able to determine all cohomology groups. For a partition λ of n, let j_λ be the corresponding irreducible G-representation.

Theorem. *Let $w = sw's \in W$ with $\mathrm{ht}(w) = 1$.*

i) If $\mathrm{ht}(sw') = 0$, then we have for $j \in \mathbb{N}$, with $\ell(w) < j < 2\ell(w) - 1$,

$$\big(H_c^{j-2}(X(w')) - j_{(j+1-n,1,\ldots,1)}(n-j)\big)(-1) - j_{(j+2-n,1\ldots,1)}(n-j-1).$$

Furthermore,

$$H_c^j(X(w)) = \begin{cases} v_B^G \bigoplus \big(v_{P(s)}^G - j_{(2,1\ldots,1)}\big)(-1) & ; \quad j = \ell(w) \\ 0 & ; \quad j = 2\ell(w) - 1 \\ i_G^G(-\ell(w)) & ; \quad j = 2\ell(w) \end{cases}.$$

ii) If $\mathrm{ht}(sw') = 1$, then we have

$$H_c^j(X(w)) = H_c^{j-2}(X(w's) \cup X(w'))(-1) \oplus H_c^{j-1}(X(sw'))$$

for all $j \neq 2\ell(w) - 1, 2\ell(w) - 2$ and $H_c^{2\ell(w)-1}(X(w)) = H_c^{2\ell(w)-2}(X(w)) = 0$.

Moreover, we give an inductive recipe for the cohomology in degree $2\ell(w) - 2$ of a DL-variety $X(w)$.

Corollary. *Let $w = sw's \in F^+$ with $\mathrm{ht}(w') \geq 1$ and $\mathrm{supp}(w) = S$. Then*

$$H_c^{2\ell(w)-2}(X(w)) = H_c^{2\ell(w's)-2}(X(w's))(-1) \bigoplus (i_{P(w')}^G - i_G^G)(-\ell(w) + 1).$$

Here $P(w')$ is the parabolic subgroup of G which is generated by B and $\mathrm{supp}(w') \subset S$.

The proof of the two last formulas bases on the second approach for determining the cohomology of DL-varieties. This alternative proposal is pursued for arbitrary elements of the Weyl group in the appendix. Whereas the previous version uses Demazure resolutions, i.e., DL-varieties attached to maximal hypersquares, this time the procedure goes the other way round in the sense that the considered hypersquare grows. In fact, for determining the cohomology of $X(w)$, we study first the map

$$H_c^i(X(w) \cup X(sw')) \longrightarrow H_c^i(X(sw'))$$

induced by the closed embedding $X(sw') \hookrightarrow X(w) \cup X(sw')$. By induction on the length the cohomology of the RHS is known. Further we give a conjecture on the structure of this map. Hence it suffices to know the cohomology of the edge $X(w) \cup X(sw')$ which is - as explained above - induced by the cohomology of the edge $X(w's) \cup X(w')$. Thus we have transferred the question of determining the cohomology of the vertex $X(w)$ to the knowledge of the cohomology of the edge $X(w's) \cup X(w')$, but which has has smaller length,

i.e. $\ell(w's) < \ell(w)$. In the next step one reduces similar the case of an edge to the case of a square etc. This second approach is a little bit vague as it depends among other things on some conjectures. Nevertheless, I have decided to include it into this paper for natural reasons.

In Section 2 we review some facts on unipotent representations of $\mathrm{GL}_n(\mathbb{F}_q)$. In Section 3 we consider DL-varieties and study explicitly the case of a Coxeter element. In Section 4 we deal with squares and their associated DL-varieties. Here we treat in particular the case of the special square $Q = \{sw's, w's, sw', w'\}$ and prove that the map $X(Q) \longrightarrow Z'$ is a $\mathbb{P}^1$-bundle. In section 5 we determine the cohomology of DL-varieties for GL_4 and in general for height 1 elements in W. In Section 6 we generalize the ideas of the forgoing section to hypersquares. Section 7 deals with the cohomology of Demazure Varieties. In Section 8 we reconsider the spectral sequence and discuss the conjecture mentioned above. Finally in Section 9 we illustrate the Conjecture resp. Theorem in the case of GL_4 .

Contents

1. Introduction 1

2. Unipotent Representations of GL_n 7

3. Deligne-Lusztig varieties 13

4. Squares 22

5. Cohomology of DL-varieties of height one 27

6. Hypersquares 34

7. Some further results on the cohomology of DL-varieties 38

8. The spectral sequence revisited 48

9. Examples 58

10. Appendix A 62

11. Appendix B 77

References 79

Notation:

- Let $k = \mathbb{F}_q$ be a finite field with fixed algebraic closure $\overline{k}$ and absolute Galois group $\Gamma := \mathrm{Gal}(\overline{k}/k)$.

- We denote for any Γ-module V and any integer i, by $V\langle i\rangle$ the eigenspace of the arithmetic Frobenius with eigenvalue q^i.

- Let $\mathbf{G_0} = \mathrm{GL}_n$ be the general linear group over k. Denote by $\mathbf{G} = \mathbf{G_0} \times_{\mathbb{F}_q} \mathbb{F}$ the base change to the algebraic closure. Let $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$ be the diagonal torus resp. the Borel subgroup of upper triangular matrices. Let $W \cong S_n$ be the Weyl group of $\mathbf{G}$ and S be the subset of simple reflections. For a subset $I \subset S$, we denote as usual by W_I be the subgroup of W generated by I.

- We also use the cyclic notation for elements in the symmetric group. Hence the expression $w = (i_1, i_2, \ldots, i_r)$ denotes the permutation with $w(i_j) = i_{j+1}$ for $j = 1, \ldots, r-1$, $w(i_r) = i_1$ and $w(i) = i$ for all $i \notin \{i_1, \ldots, i_r\}$.

- For a vector space V of dimension n over k and any integer $1 \leq i \leq n$, we let $\mathrm{Gr}_i(V)$ be the Grassmannian parametrizing subspaces of dimension i.

- We denote by 1 or e the identity in any group or monoid.

Acknowledgements: I thank Roland Huber, Michael Rapoport and Markus Reineke for helpful discussions.

2. Unipotent Representations of GL_n

We start with discussing representations of $G = \mathbf{GL_n}(k)$ which are called unipotent. These kind of objects appear in the cohomology of Deligne-Lusztig varieties. In this section, all representations will be in vector spaces over a fixed algebraically closed field C of characteristic zero.

Recall that a standard parabolic subgroup (std psgp) $\mathbf{P} \subset \mathbf{G}$ is a parabolic subgroup of $\mathbf{G}$ with $\mathbf{B} \subset \mathbf{P}$. The set of all std psgp is in bijection with the set

$$\mathcal{D} = \mathcal{D}(n) = \big\{(n_1, \ldots, n_r) \in \mathbb{N}^r \mid n_1 + \cdots + n_r = n,\, r \in \mathbb{N}\big\}$$

of decompositions of n. For a decomposition $d = (n_1, \ldots, n_r) \in \mathcal{D}(n)$, we let $\mathbf{P}_d$ be the corresponding std psgp with Levi subgroup $\mathbf{M_{P_d}} = \prod_i \mathbf{GL}_{n_i}$. There is a partial order $\leq$ on $\mathcal{D}(n)$ defined by

$$d_1 \leq d_2 \text{ if and only if } \mathbf{P}_{d_1} \subset \mathbf{P}_{d_2}.$$

If $\mathbf{P} = \mathbf{P}_d$ is a std psgp to a decomposition $d \in \mathcal{D}$, then any $d' \in \mathcal{D}$ induces a std psgp $\mathbf{Q}_{d'} = \mathbf{M_P} \cap \mathbf{P}_{d'}$ of $\mathbf{M_P}$ and this assignment gives a bijection between the sets

$$\{d' \in \mathcal{D} \mid d' \leq d\} \xrightarrow{\sim} \{\text{ std psgps of } \mathbf{M_P}\}.$$

In the sequel we call the finite group $P = \mathbf{P}(k)$ attached to a std psgp $\mathbf{P}$ standard parabolic, as well.

Recall that a parabolic subgroup of W is by definition the Weyl group of the Levi component $\mathbf{M_P}$ of some std psgp $\mathbf{P}$. This defines a one-to-one correspondence between the std psgp $\mathbf{P}$ of $\mathbf{G}$ and the parabolic subgroups W_P of W. If $\mathbf{P} = \mathbf{P}_d$ with $d = (n_1, \ldots, n_r) \in \mathcal{D}$, then $W_P = S_{n_1} \times \cdots \times S_{n_r}$.

We denote by $\mathcal{P} = \mathcal{P}(n)$ the set of partitions of n. There is a map $\lambda : \mathcal{D} \longrightarrow \mathcal{P}$ by ordering a decomposition $(n_1, \ldots, n_r)$ in decreasing size. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\mu' = (\mu'_1, \mu'_2, \ldots, \mu'_n)$ be two partitions of n. Consider the order $\leq$ on $\mathcal{P}$ defined by $\mu' \leq \mu$ if for all $r = 1, \ldots, n$, we have

$$\sum_{i=1}^{r} \mu'_i \leq \sum_{i=1}^{r} \mu_i.$$

Then the map λ is compatible with both orders. If $P = P_d$ is a std psgp, then we also write $\lambda(P)$ for $\lambda(d)$.

On the other hand, two parabolic subgroups are called associate, if their Levi components are conjugate under G. If $P = P_{d_1}$ and $P' = P_{d_2}$ with associated decompositions $d_1 = (n_1, \ldots, n_r)$ and $d_2 = (n'_1, \ldots, n'_{r'})$ of n, then P and P' are associate if and only if $\lambda(d_1) = \lambda(d_2)$.

Let P be a std psgp of G. We denote by

$$i_P^G = \mathrm{Ind}_P^G \mathbf{1}$$

the induced representation of the trivial representation $\mathbf{1}$ of P. Let $\hat{G}(i_B^G)$ be the set of isomorphism classes of irreducible subobjects of i_B^G. We remind the reader at the following properties of the representations i_P^G, cf. [DOR, Thm. 3.2.1].

Theorem 2.1. *(i) i_P^G is equivalent to $i_{P'}^G$ if and only if P is associate to P'.*

(ii) i_P^G contains a unique irreducible subrepresentation j_P^G which occurs with multiplicity one and such that

$$\mathrm{Hom}_G(j_P^G, i_{P'}^G) \neq (0) \Leftrightarrow \lambda(P') \leq \lambda(P).$$

(iii) We set for every $\mu \in \mathcal{P}$,

$$j_\mu = j_{P_\mu}^G$$

where P_μ is any std psgp with $\lambda(P_\mu) = \mu$. Then $\{j_\mu \mid \mu \in \mathcal{P}\}$ is a set of representatives for $\hat{G}(i_B^G)$.

Remark 2.2. The proof of the above theorem makes use of the representation theory of the symmetric group and bases on the following result of Howe [Ho]. Let $\hat{W}$ be the set of

isomorphism classes of irreducible representations of W. Analogously to the definition of i_P^G, we set $i_{W'}^W := \mathrm{Ind}_{W'}^W \mathbf{1}$ for any subgroup W' of W. There exists a unique bijection

$$\alpha : \hat{G}(i_B^G) \longrightarrow \hat{W}$$

characterized by the following property. An irreducible representation $\sigma \in \hat{G}(i_B^G)$ occurs in i_P^G if and only if $\alpha(\sigma)$ occurs in $i_{W_P}^W$. Furthermore

$$\dim \mathrm{Hom}_G(\sigma, i_P^G) = \dim \mathrm{Hom}_W(\alpha(\sigma), i_{W_P}^W).$$

In particular, we get by Frobenius reciprocity

$$\mathrm{Hom}_G(i_Q^G, i_P^G) \cong \mathrm{Hom}_W(i_{W_Q}^W, i_{W_P}^W) = \mathrm{Hom}_{W_Q}(\mathbf{1}, i_{W_P}^W) = C[W_Q \backslash W / W_P].$$

Let $P \subset Q$ be two standard parabolic subgroups. Then there is a natural inclusion of G-representations $i_Q^G \subset i_P^G$ which corresponds just to the double coset of $e \in W$ in $W_Q \backslash W / W_P$, cf. the remark above.

Definition 2.3. The *generalized Steinberg representation* associated to a std psgp $P = P_d$ is the quotient

$$v_P^G = i_P^G / \sum_{Q \supsetneq P} i_Q^G = i_{P_d}^G / \sum_{d' > d} i_{P_{d'}}^G.$$

For $P = G$, we have $v_G^G = \mathbf{1}$ whereas if $B = P$ then we get the ordinary Steinberg representation v_B^G which is irreducible.

Remark 2.4. In general, the generalized Steinberg representations v_P^G are not irreducible. Indeed it follows from Remark 2.2 and the that $v_{P_d}^G$ is irreducible if and only if $d = (k, 1, \ldots, 1)$ for some k with $1 \leq k \leq n$.

Let us recall some further property of the induced representations i_P^G. Let $P = P_d \subset G$ for some $d \in \mathcal{D}$. Let $d' \leq d$ and consider the std psgp $Q_{d'} \subset M_P$ of M_P. We consider $i_{Q_{d'}}^{M_P}$ as a P-module via the trivial action of the unipotent radical of P. Then

$$\mathrm{Ind}_P^G(i_{Q_{d'}}^{M_P}) = i_{P_{d'}}^G.$$

Since Ind_P^G is an exact functor we get for any $d'' \in \mathcal{D}$ with $d' \leq d'' \leq d$, the identity $\mathrm{Ind}_P^G(i_{Q_{d'}}^{M_P} / i_{Q_{d''}}^{M_P}) = i_{P_{d'}}^G / i_{P_{d''}}^G$. In particular, we conclude that

$$\mathrm{Ind}_P^G(v_{Q_{d'}}^{M_P}) = i_{P_{d'}}^G / \sum_{\{d'' \in \mathcal{D} \mid d' < d'' \leq d\}} i_{P_{d''}}^G.$$

Consider the special situation where $d = (n_1, n_2) \in \mathcal{D}$. Then $P = P_{(n_1, n_2)}$ and $M_P = M_1 \times M_2$ with $M_1 = \mathrm{GL}_{n_1}$, $M_2 = \mathrm{GL}_{n_2}$. Let for $i = 1, 2$, $d_i \in \mathcal{D}(n_i)$ be a decomposition

of n_i and consider the corresponding std psgps P_{d_i} of M_i. Denote by $(d_1, d_2) \in \mathcal{D}(n)$ the glued decomposition of n.

Lemma 2.5. *We have the identity*

$$\mathrm{Ind}^G_{P_{(n_1,n_2)}}(v^{M_1}_{P_{d_1}} \boxtimes v^{M_2}_{P_{d_2}}) = i^G_{P_{(d_1,d_2)}} / \sum_{d_1' > d_1} i^G_{P_{(d_1',d_2)}} + \sum_{d_2' > d_2} i^G_{P_{(d_1,d_2')}}. \tag{2.1}$$

Proof. Since

$$i^{M_1}_{P_{d_1}} \boxtimes i^{M_2}_{P_{d_2}} = i^{M_1 \times M_2}_{P_{d_1} \times P_{d_2}}$$

we get

$$\mathrm{Ind}^G_{P_{(n_1,n_2)}}(i^{M_1}_{P_{d_1}} \boxtimes i^{M_2}_{P_{d_2}}) = i^G_{P_{(d_1,d_2)}}.$$

Then the identity above follows by applying the exact functor $\mathrm{Ind}^G_{P_{(n_1,n_2)}}$ to the exact sequence

$$0 \longrightarrow \sum_{d_1' > d_1} i^{M_1}_{P_{d_1'}} \boxtimes i^{M_2}_{P_{d_2}} + \sum_{d_2' > d_2} i^{M_1}_{P_{d_1}} \boxtimes i^{M_2}_{P_{d_2'}} \longrightarrow i^{M_1}_{P_{d_1}} \boxtimes i^{M_2}_{P_{d_2}} \longrightarrow v^{M_1}_{P_{d_1}} \boxtimes v^{M_2}_{P_{d_2}} \longrightarrow 0.$$

$\square$

For the next property of generalized Steinberg representations, we refer to [Le] (resp. to [DOR] for a detailed discussion on this complex). Here we set for any $d = (n_1, \ldots, n_r) \in \mathcal{D}$, $r(d) = r$.

Proposition 2.6. *Let $P = P_d$, where $d \in \mathcal{D}$. Then there is an acyclic resolution of v^G_P by G-modules,*

$$0 \to i^G_G \to \bigoplus_{\substack{d' \geq d \\ r(d')=2}} i^G_{P_{d'}} \to \bigoplus_{\substack{d' \geq d \\ r(d')=3}} i^G_{P_{d'}} \to \cdots \to \bigoplus_{\substack{d' \geq d \\ r(d')=r(d)-1}} i^G_{P_{d'}} \to i^G_{P_d} \to v^G_P \to 0. \tag{2.2}$$

Remark 2.7. The prove of Proposition 2.6 relies on some simplicial arguments as follows, cf. loc.cit. Let $d, d' \in \mathcal{D}$. Then we have

$$i^G_{P_d} \ \cap \ i^G_{P_{d'}} = i^G_{P_{d \vee d'}} \tag{2.3}$$

for some $d \vee d' \in \mathcal{D}$. Further for all $d_1, \ldots, d_r \in \mathcal{D}$, we have

$$i^G_{P_d} \ \cap \ (i^G_{P_{d_1}} + i^G_{P_{d_2}} + \cdots + i^G_{P_{d_r}}) = (i^G_{P_d} \cap i^G_{P_{d_1}}) \ + \ \cdots + (i^G_{P_d} \cap i^G_{P_{d_r}}). \tag{2.4}$$

We reinterpret this complex as follows. Let F^+ be the monoid which is freely generated by the subset $S \subset W$ of simple reflections. Denote by

$$\gamma : F^+ \longrightarrow W$$

the natural map. For $w = s_{i_1} \cdots s_{i_r} \in F^+$, let $r = \ell(w)$ be the length and

$$\mathrm{supp}(w) = \{s_{i_1}, \ldots, s_{i_r}\}$$

its support. Any subword v which is induced by erasing factors in w gives by definition rise to an element which is shorter in the Bruhat ordering $\preceq$ on F^+, cf. also [Hu]. Note that this ordering is not compatible with the usual one $\leq$ on W via γ.

For $w \in F^+$, let $I(w) \subset S$ be a minimal subset such that w is contained in the submonoid generated by $I(w)$. Let

$$P(w) = P_{I(w)} \subset G \tag{2.5}$$

be the std parabolic subgroup generated by B and $I(w)$. Alternatively, let $d(w) \in \mathcal{D}$ be the decomposition which corresponds to the subset $\mathrm{supp}(w) \subset S$ under the natural bijection $\mathcal{D} \xrightarrow{\sim} S$. Then $P(w) = P_{d(w)}$.

We may define for $w \in F^+$ the following complex where the differentials are defined similar as above:

$$C_w^\bullet : 0 \to i^G_{P(w)} \to \bigoplus_{\substack{v \preceq w \\ \ell(v)=\ell(w)-1}} i^G_{P(v)} \to \bigoplus_{\substack{v \preceq w \\ \ell(v)=\ell(w)-2}} i^G_{P(v)} \to \cdots \to \bigoplus_{\substack{v \preceq w \\ \ell(v)=1}} i^G_{P(v)} \to i^G_{P(e)} \to 0. \tag{2.6}$$

Example 2.8. Let $w = \mathrm{Cox}_n = s_1 s_2 \ldots s_{n-1} \in F^+$ be the standard Coxeter element. Then the complex $C_w^\bullet$ coincides - up to augmentation with respect to the Steinberg representation - with the complex (2.2) where $d = d(e)$.

Definition 2.9. Let $w \in F^+$. Then we say that w has full support if $\mathrm{supp}(w) = S$.

Proposition 2.10. *Let $w \in F^+$ have full support. Then the complex $C_w^\bullet$ is quasi-isomorphic to $C_{\mathrm{Cox}}^\bullet$.*

Proof. We may suppose that w is not a Coxeter element. Hence there exist a simple reflection $s \in S$ which appears at least twice in w. Write $w = w_1 s w' s w_2$ with $s \in S$ and $w_1, w_2, w' \in F^+$. Of course the subword $v = w_1 w' s w_2$ has full support, as well. Hence by induction the complex $C_v^\bullet$ is quasi-isomorphic to $C_{\mathrm{Cox}}^\bullet$. On the other hand, for any subword $v_1 s v' s v_2$ of w, we have $P(v_1 s v' v_2) = P(v_1 s v' s v_2)$. Hence the difference between the complexes $C_w^\bullet$ and $C_v^\bullet$ is a contractible complex. The result follows. $\square$

Let $w \in F^+$ and $s \in \mathrm{supp}(w)$. Hence we may write $w = w_1 s w_2$ for some subwords $w_1, w_2 \in F^+$ of w. Then we also use the notation

$$w/s := w_1 w_2 \in F^+$$

for convenience. We may define analogously to (2.6) the complex

$$C^\bullet_{w,s} : 0 \to i^G_{P(w)} \to \bigoplus_{\substack{s \preceq v \preceq w \\ \ell(v)=\ell(w)-1}} i^G_{P(v)} \to \bigoplus_{\substack{s \preceq v \preceq w \\ \ell(v)=\ell(w)-2}} i^G_{P(v)} \to \cdots \to i^G_{P(s)} \to 0. \tag{2.7}$$

Example 2.11. Let $w = \mathrm{Cox}_n \in F^+$ be the standard Coxeter element. Then the complex $C^\bullet_{w,s}$ coincides - up to augmentation with respect to the generalized Steinberg representation - with the complex (2.2) where $d = d(s)$, i.e., $P = P(s)$.

With the same arguments as in Proposition 2.10 one proves the next statement.

Proposition 2.12. *Let $w \in F^+$ have full support and let $s \in \mathrm{supp}(w)$. Then the complex $C^\bullet_{w,s}$ is quasi-isomorphic to $C^\bullet_{\mathrm{Cox},s}$ if $s \notin \mathrm{supp}(w/s)$. Otherwise, it is acyclic.* □

More generally, fix $w \in F^+$ and a subword $u \prec w$. For any $v \in F^+$ with $u \preceq v \preceq w$, let $P_v \subset G$ be a std psgp chosen inductively in the following way. Choose an arbitrary std psgp P_u. Let $u \prec v \preceq w$ and suppose that for all $u \preceq z \prec v$, the std psgp P_z are already defined. Set $\{z_1, \ldots, z_r\} = \{u \preceq z \prec v \mid \ell(z) = \ell(v) - 1\}$. Then let $P_v \subset G$ be a std psgp such that $i^G_{P_v} = A \oplus B$ where $A \subset \bigcap_i i^G_{P_{z_i}}$ and B maps to zero under all the maps $i^G_{P_v} \longrightarrow i^G_{P_{z_i}}$ induced by the double cosets of $e \in W$ in $W_{P_v} \setminus W / W_{P_{z_i}}$ via Frobenius reciprocity. Hence we get a sequence of G-representations

$$0 \to i^G_{P_w} \to \bigoplus_{\substack{u \preceq v \preceq w \\ \ell(v)=\ell(w)-1}} i^G_{P_v} \to \bigoplus_{\substack{u \preceq v \preceq w \\ \ell(v)=\ell(w)-2}} i^G_{P_v} \to \cdots \to \bigoplus_{\substack{u \preceq v \preceq w \\ \ell(v)=\ell(u)+1}} i^G_{P_v} \to i^G_{P_u} \to 0 \tag{2.8}$$

which we equip analogously with the same signs as above. By the very construction of the sequence we derive the following fact.

Lemma 2.13. *The sequence* (2.8) *is a complex.* □

Example 2.14. Let $\mathbf{G} = \mathrm{GL}_4$ and $w, u \in F^+$ with $\ell(w) = \ell(u) + 2$. The sequence

$$i^G_{P_{(3,1)}} \longrightarrow i^G_{P_{(3,1)}} \oplus i^G_{P_{(2,2)}} \longrightarrow i^G_{P_{(3,1)}}$$

(with differentials as explained above) is a complex, whereas

$$i^G_{P_{(2,2)}} \longrightarrow i^G_{P_{(2,2)}} \oplus i^G_{P_{(3,1)}} \longrightarrow i^G_{P_{(2,2)}}$$

is not.

For later use, we introduce the next definition.

Definition 2.15. Let V be a finite-dimensional G-representation. We denote by $\mathrm{supp}(V)$ the set of isomorphism classes of irreducible subrepresentations which appear in V.

Let $f : V \longrightarrow W$ be a homomorphism of G-representations. We get for each irreducible G-representation Z an induced map

$$f^Z : V^Z \longrightarrow W^Z$$

of the Z-isotypic parts.

Definition 2.16. Let $f : V \longrightarrow W$ be a homomorphism of G-representations.

i) We call f si-surjective resp. si-injective resp. si-bijective if the map f^Z is surjective resp. injective resp. bijective for all $Z \in \mathrm{supp}(V) \cap \mathrm{supp}(W)$.

ii) We say that f has si-full rang if the map f^Z has full rang for all $Z \in \mathrm{supp}(V) \cap \mathrm{supp}(W)$.

Remarks 2.17. i) The above definition makes of course sense for arbitrary groups. We will apply it in the upcoming sections in the case of $H := G \times \Gamma$. Here we shall see later on that the action of H on the considered geometric representations is semi-simple.

ii) Obviously, the homomorphism f has si-full rang if and only if the map f^Z is injective or surjective for all $Z \in \mathrm{supp}(V) \cap \mathrm{supp}(W)$.

We close this section with the following observation.

Lemma 2.18. *Let* $V^1 \stackrel{f_1}{\longrightarrow} V^2 \stackrel{f_2}{\longrightarrow} V^3 \stackrel{f_3}{\longrightarrow} V^4$ *be an exact sequence of* H*-modules with* $\mathrm{supp}(V^1) \cap \mathrm{supp}(V^4) = \emptyset$.

a) If f_1 *is si-surjective then* f_2 *is si-injective.*

b) If f_3 *si-injective then* f_2 *si-surjective,*

c) f_2 *has si-full rang.*

Proof. a) Let $Z \in \mathrm{supp}(V^2) \cap \mathrm{supp}(V^3)$. If the map f_2^Z has a kernel it follows that $V \in \mathrm{supp}(V^1)$. Since f_1 is si-surjective we deduce that f_2^Z is the zero map. Hence V_3^Z maps injectively into V^4 which implies $Z \in \mathrm{supp}(V^4)$ which is a contradiction to the assumption.

b) Let $Z \in \mathrm{supp}(V^2) \cap \mathrm{supp}(V^3)$. If the map f_2^Z is not surjective it follows that $V \in \mathrm{supp}(V^4)$. Since f_3 is si-injective we deduce that f_2^Z is the zero map which implies $Z \in \mathrm{supp}(V^1)$ which is again a contradiction to the assumption.

c) This is obvious. □

3. Deligne-Lusztig varieties

Let $X = X_{\mathbf{G}}$ be the set of all Borel subgroups of $\mathbf{G}$. This is a smooth projective algebraic variety homogeneous under $\mathbf{G}$. By the Bruhat lemma the set of orbits of $\mathbf{G}$ on $X \times X$ can

be identified with W. We denote by $\emptyset(w)$ the orbit of $(B, wBw^{-1}) \subset X \times X$ and by $\overline{\emptyset(w)} \subset X \times X$ its Zariski closure.

Let $F : X \to X$ be the Frobenius map over $\mathbb{F}_q$. The *Deligne-Lusztig variety associated to* $w \in W$ is the locally closed subset of X given by

$$X(w) = X_{\mathbf{G}}(w) = \{x \in X \mid \mathrm{inv}(x, F(x)) = w\}$$

where by definition $\mathrm{inv}(x, F(x)) = w \Leftrightarrow (x, F(x) \in \emptyset(w)$. Denote by $\leq$ the Bruhat order and by ℓ the length function on W. Then $X(w)$ is a smooth quasi-projective variety of dimension $\ell(w)$ defined over $\mathbb{F}_q$ and which is equipped with an action of G, cf. [DL, 1.4]. We denote by $\overline{X(w)} \subset X$ its Zariski closure.

Before we proceed, let us recall some properties of the varieties $\emptyset(w)$ we need in the sequel, cf. loc.cit. If $w = w_1 w_2$ with $\ell(w) = \ell(w_1) + \ell(w_2)$ then

(1) a) $(B, B') \in \emptyset(w_1)$ and $(B', B'') \in \emptyset(w_2)$ implies $(B, B'') \in \emptyset(w)$

b) If $(B, B'') \in \emptyset(w)$, then there is a unique $B' \in X$ with $(B, B') \in \emptyset(w_1)$ and $(B', B'') \in \emptyset(w_2)$.

In other words, there is an isomorphism of schemes $\emptyset(w_1) \times_X \emptyset(w_2) \cong \emptyset(w)$.

(2) Let $w, w' \in W$. Then $\emptyset(w') \subset \overline{\emptyset(w)} \Leftrightarrow w' \leq w$ for the Bruhat order $\leq$ on W.

As in the case of usual Schubert cells, there is by item (2) the following relation concerning the closures of DL-varieties. We have

$$X(w') \subset \overline{X(w)} \Leftrightarrow w' \leq w.$$

It follows that we have a Schubert type stratification

$$\overline{X(w)} = \bigcup_{v \leq w} X(v). \tag{3.1}$$

In particular if $w' \leq w$ with $\ell(w) = \ell(w') + 1$, then $X(w) \cup X(w')$ is a locally closed subvariety of X which is moreover smooth since the dimensions of $X(w)$ and $X(w')$ differ by one.

Example 3.1. Let $G = \mathrm{GL}_3$ and identify X with the full flag variety of $V = \mathbb{F}^3$. Then

$$\begin{aligned}
X(1) &= X(\mathbb{F}_q) \\
X((1,2)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid V^2 \text{ is } k\text{-rational}, F(V^1) \neq V^1\} \\
X((2,3)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid V^1 \text{ is } k\text{-rational}, F(V^2) \neq V^2\} \\
X((1,2,3)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid F(V^1) \subset V^2, F(V^i) \neq V^i, i = 1,2\} \\
X((1,3,2)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid V^1 \subset F(V^2), F(V^i) \neq V^i, i = 1,2\} \\
X((1,3)) &= \{(0) \subset V^1 \subset V^2 \subset V \mid F(V^1) \not\subseteq V^2, V^1 \not\subseteq F(V^2)\}.
\end{aligned}$$

Let $S = \{s_1, \ldots, s_{n-1}\} \subset W$ be the set of simple reflections. Recall that a Coxeter element $w \in W$ is given by any product of all $s \in S$ (with multiplicity one). The Coxeter number h is the length of any Coxeter element, i.e. $h = n - 1$. In the sequel we denote by

$$\mathrm{Cox}_n := s_1 \cdot s_2 \cdots s_h = (1, 2, \ldots, n) \in W$$

the standard Coxeter element.

Example 3.2. Let $w = \mathrm{Cox}_n$. Then $X(w)$ can be identified via the projection map $X \longrightarrow \mathbb{P}^{n-1}$ with the Drinfeld space

$$\Omega(V) = \Omega^n = \mathbb{P}^{n-1} \setminus \bigcup\nolimits_{H/\mathbb{F}_q} H$$

(complement of all $\mathbb{F}_q$-rational hyperplanes in the projective space of lines in $V = \mathbb{F}^n$), cf. [DL], §2. Its inverse is given by the map

$$\begin{aligned}
\Omega(V) &\longrightarrow X(w) \\
x &\mapsto x \subset x + F(x) \subset x + F(x) + F^2(x) \subset \cdots \subset V
\end{aligned}$$

For any Coxeter element w for GL_n, the corresponding DL-variety $X(w)$ is universally homeomorphic to Ω^n, cf. [L2], Prop. 1.10.

In the sequel we denote for any variety X defined over k by $H^i_c(X) = H^i_c(X, \overline{\mathbb{Q}}_\ell)$ (resp. $H^i(X) = H^i(X, \overline{\mathbb{Q}}_\ell)$) the ℓ-adic cohomology with compact support (resp. the ℓ-adic cohomology) in degree i. For a Deligne-Lusztig variety $X(w)$, there is by functoriality an action of $H = G \times \Gamma$ on these cohomology groups.

Proposition 3.3. *Let $w = \mathrm{Cox}_n$ be the standard Coxeter element. Then*

$$H^*_c(X(w)) = \bigoplus_{k=1,\ldots,n} j_{(k,1,\ldots,1)}(-(k-1))[-(n-1)-(k-1)].$$

(Here for an integer $m \in \mathbb{Z}$, we denote as usual by (m) the Tate twist of degree m.)

Proof. See [L2, O] resp. [SS] in the case of a local field. □

The cohomology of the Zariski closure of the DL-variety $X(\mathrm{Cox}_n)$ has the following description.

Proposition 3.4. *Let $w = \mathrm{Cox}_n \in W$ be the standard Coxeter element. Then*

$$H^*(\overline{X(w)}) = \bigoplus_{v \leq w} J(v)(-\ell(v))[-2\ell(v)],$$

where $J(v) = J^G(v)$ is defined inductively as follows. Write v in the shape

$$v = v' \cdot s_{j+1} \cdot s_{j+2} \cdots s_{j+l}$$

where $1 \leq j \leq n-2$, $l \geq 0$ and where $v' \leq \mathrm{Cox}_j = s_1 \cdots s_{j-1}$. Then

$$J(v) = \mathrm{Ind}^G_{P_{(j,n-j)}}(J^{\mathrm{GL}_j}(v') \boxtimes \mathbf{1}).$$

Proof. In terms of flags the DL-variety $X(w)$ has the description

$$X(w) \quad = \quad \{V^\bullet \mid F(V^j) \subset V^{j+1}, F(V^j) \neq V^j, \; \forall\, 1 \leq j \leq n-1\}$$

The Zariski closure $\overline{X(w)}$ of $X(w)$ in X is then given by the subset

$$\overline{X(w)} \quad = \quad \{V^\bullet \mid F(V^j) \subset V^{j+1}, \; \forall\, 1 \leq j \leq n-1\}$$

which can be identified with a sequence of blow-ups as follows, cf. [Ge, GK, I]. Start with $Y_0 = \mathbb{P}^{n-1} = \mathbb{P}(V)$ where $V = \mathbb{F}^n$ and consider the blow up B_1 in the set of rational points $Z_0 = \mathbb{P}^{n-1}(k) \subset Y_0$. Then we may identify B_1 with the variety $\{V^1 \subset V^2 \subset V \mid F(V^1) \subset V^2\}$. We set $Y_1 = \bigcup_{W \in \mathrm{Gr}_2(V)(k)} \mathbb{P}(W) \subset B_1$ which is the strict transform of the finite set of k-rational planes and blow up B_1 in Y_1. The resulting variety B_2 can be identified with $\{V^1 \subset V^2 \subset V^3 \subset V \mid F(V^i) \subset V^{i+1}, i = 1, 2\}$. Now we repeat this construction successively until we get $B_{n-2} = \overline{X(w)}$. Hence the cohomology of $\overline{X(w)}$ can be deduced from the usual formula for blow ups [SGA5]. More precisely, each time we blow up, we have to add the cohomology of the variety

$$\coprod_{W \in \mathrm{Gr}_j(V)(k)} \overline{X_{\mathrm{GL}_j}(\mathrm{Cox}_j)} \times \mathbb{P}(V/W) \setminus \overline{X_{\mathrm{GL}_j}(\mathrm{Cox}_j)} \times \mathbb{P}^0.$$

The start of this procedure is given by the cohomology of the projective space $\mathbb{P}(V)$, which we initialize by $H^{2i}(\mathbb{P}(V)) = J(s_1 s_2 \cdots s_i)(-i) = i^G_G(-i), i = 1, \ldots, n-1$. □

Remarks 3.5. i) Since all DL-varieties for Coxeter elements are homeomorphic [L2], they have all the same cohomology. By considering stratifications (3.1) for different Coxeter elements, the same is true for their Zariski closures. Alternatively, one might argue that

the morphism σ, τ of Proposition 4.2 for different Coxeter elements extend to their Zariski closures, thus inducing an isomorphism on their cohomology.

ii) It follows by the description of $\overline{X}(\mathrm{Cox}_n)$ in terms of blow ups together with the remark before that thy cycle map

$$A^i(\overline{X}(w))_{\overline{\mathbb{Q}}_\ell} \longrightarrow H^{2i}(\overline{X}(w))$$

is an isomorphism for every Coxeter element w. In fact, in the Chow group the same formulas concerning blow ups are valid [SGA5].

iii) Let $w = s_h s_{h-1} \cdots s_2 s_1$. Then by symmetry, there is is by considering the dual projective space $(\mathbb{P}^{n-1})^\vee$ a procedure for realising

$$\overline{X(w)} = \{V^\bullet \mid V^i \subset F(V^{i+1}),\ i = 1, \ldots, n-1\}$$

as a sequence of blow ups.

On the other hand, for elements $w \in W$ having not full support, the cohomology of $X(w)$ can be deduced by an induction process. Here recall that we say that w has full support if it not contained in any proper parabolic subgroup W_P of W.

Proposition 3.6. *Let $w \in W$ have not full support. Let $\mathbf{P} = \mathbf{P}_{(\mathbf{i_1}, \ldots, \mathbf{i_r})}$ be a minimal parabolic subgroup such that $w \in W_P$. Then*

$$H_c^*(X(w)) = \mathrm{Ind}_P^G(H_c^*(X_{\mathbf{M_P}}(w))),$$

where $X_{\mathbf{M_P}}(w) = \prod_{j=1}^r X_{\mathrm{GL}_{i_j}}(w_j)$ and $w = w_1 \cdots w_r$ with $w_j \in S_{i_j}, j = 1, \ldots, r$.

Proof. See [DL, Prop. 8.2]. □

Remark 3.7. The same formula holds true with respect to the Zariski closures $\overline{X(w)}$ of elements having not full support, as is the case for Chow groups. In particular, the cycle map is an isomorphism for all those elements, as well.

In the case when we erase some simple reflection in a reduced expression of a Coxeter element, we may deduce from the above results the following consequence. Here we abbreviate the partition or decomposition $(k, 1, \ldots, 1) \in \mathcal{P}$ by $(k, 1^{(n-k)})$. For any triple of integers $i, k, l \in \mathbb{Z}$ with $1 \leq i \leq n$ and $1 \leq k \leq i$, $1 \leq l \leq n-i$, we set

$$A_{k,l} = i^G_{P_{(k,1^{(i-k)},l,1^{(n-i-l)})}} / \sum_{d_1 > (k,1^{(i-k)})} i^G_{P_{(d_1,l,1^{(n-i-l)})}} + \sum_{d_2 > (l,1^{(n-i-l)})} i^G_{P_{(k,1^{(k-i)},d_2)}}.$$

Corollary 3.8. *Let $w = \mathrm{Cox}_n = s_1 \cdots s_h$ be the standard Coxeter element and let*[2] $w' = s_1 \cdots \hat{s}_i \cdots s_h \in W$. *Then*

$$H_c^*(X(w')) = \bigoplus_{m=2}^{n} \bigoplus_{k+l=m} A_{k,l}(-(m-2))[-(n-2)-(m-2)].$$

Proof. The Weyl group element w' is contained in the parabolic subgroup W_P with $P = P_{(i,n-i)}$. Now the expressions $s_1 \cdots s_{i-1}$ and $s_{i+1} \cdots s_h$ are both Coxeter elements in the Weyl groups attached to $M_1 = \mathrm{GL}_i$ resp. $M_2 = \mathrm{GL}_{n-i}$. The cohomology of $H_c^*(X_{M_1}(w_1))$ resp. $H_c^*(X_{M_2}(w_2))$ are given by Proposition 3.3:

$$H_c^*(X_{M_1}(w_1)) = \bigoplus_{k=1,\ldots,i} j_{(k,1,\ldots,1)}^{M_1}(-(k-1))[-(i-1)-(k-1)]$$

resp.

$$H_c^*(X_{M_2}(w_2)) = \bigoplus_{l=1,\ldots,n-i} j_{(l,1,\ldots,1)}^{M_2}(-(l-1))[-((n-i)-1)-(l-1)].$$

Thus we get by Proposition 3.6 and identity (2.1)

$$\begin{aligned} H_c^*(X(w')) &= \mathrm{Ind}_{P_{(i,n-i)}}^G \Big(H_c^*(X_{M_1}(w_1)) \boxtimes H_c^*(X_{M_2}(w_2)) \Big) \\ &= \bigoplus_{m=2}^{n} \bigoplus_{k+l=m} A_{k,l}(-(k+l-2))[-(n-2)-(k+l-2)]. \end{aligned}$$

□

Let $w = \mathrm{Cox}_n \in W$. As we see from Proposition 3.4 the cohomology of $\overline{X}(w)$ vanishes in odd degree. For any integer $i \geq 0$, let

$$H^{2i}(\overline{X(w)}) \longrightarrow \bigoplus_{\substack{v<w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X(v)}) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{v<w \\ \ell(v)=1}} H^{2i}(\overline{X(v)}) \longrightarrow H^{2i}(\overline{X(e)}) \tag{3.2}$$

be the natural complex induced by the closed complement $\bigcup_{v<w} \overline{X(v)}$ of $X(w)$ in $\overline{X(w)}$. This complex determines the contribution with Tate twist $-i$ to the cohomology of $X(w)$.

On the other hand, we may consider the grading

$$H^{2i}(\overline{X(w)}) = \bigoplus_{\substack{z \leq w \\ \ell(z)=i}} H(w)_z$$

described in Proposition 3.4, i.e. $H(w)_z = J(z)(-\ell(z))$ for $z < w$. By Proposition 3.6 we also have such a grading for all subexpressions $v < w$, i.e. $H^{2i}(\overline{X(v)}) = \bigoplus_{\substack{z \leq v \\ \ell(z)=i}} H(v)_z$.

[2] Here the symbol $\hat{s}_i$ means as usual that s_i is deleted from the above expression.

Moreover $H(w)_z \subset H(v)_z$ for all $z < v$. Hence we get a graded complex

$$(3.3)\quad \bigoplus_{\substack{z\leq w\\ \ell(z)=i}} H(w)_z \longrightarrow \bigoplus_{\substack{v\leq w\\ \ell(v)=\ell(w)-1}} \bigoplus_{\substack{z\leq v\\ \ell(z)=i}} H(v)_z \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{v\leq w\\ \ell(v)=1}} \bigoplus_{\substack{z\leq v\\ \ell(z)=i}} H(v)_z \longrightarrow \bigoplus_{\substack{z\leq e\\ \ell(z)=i}} H(e)_z$$

$$= \bigoplus_{\substack{z\leq w\\ \ell(z)=i}} \Big(H(w)_z \longrightarrow \bigoplus_{\substack{z\leq v\leq w\\ \ell(v)=\ell(w)-1}} H(v)_z \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{z\leq v\leq w\\ \ell(v)=1}} H(v)_z \longrightarrow \bigoplus_{\substack{z\leq e\\ \ell(z)=i}} H(e)_z \Big).$$

It is a direct sum of complexes of type (2.8).

Proposition 3.9. *The complexes (3.2) and (3.3) are quasi-isomorphic.*

Proof. The idea is to show that the differentials in (3.2) are in triangular form with respect to the gradings. By induction on n it suffices to consider the first differential

$$H^{2i}(\overline{X(w)}) \longrightarrow \bigoplus_{\substack{v\prec w\\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X(v)}).$$

Here the result follows from the description of the cohomology of $\overline{X(w)}$ in Proposition 3.4 via blow ups. In fact, every contribution $H(w)_z = J(z)(-\ell(z))$ with $z = v'\cdot s_{j+1}\cdot s_{j+2}\cdots s_{j+l}$ and $j \geq 1$ maps isomorphically - by the blow up formula - onto its image $H(v)_z$ in $H^*(\overline{X(v)})$ where $v = s_1\cdots s_{j-1}\cdot s_{j+1}\cdots s_{n-1}$. Further it can intertwine only with a contribution $H(v')_{z'}$, with $z' = v'' s_{k+1}\cdots s_{k+l}$ where $k < j$. As for the remaining contributions $J(z)$ with $z \in \{e, s_1, s_1\cdot s_2, \ldots, \mathrm{Cox}\}$ we have $H(w)_z = J(z)(-\ell(z)) = i_G^G(-\ell(z))$. They do not intertwine with other $H(v)_{z'}$ for $z' \neq z$. Moreover, the complexes attached to these latter elements just induce the cohomology of $X(w)$. The result follows. □

Conjecture 3.10. *More generally, let $u < w$ and $i \in \mathbb{N}$. Then for all $u < v < w$, there are gradings*

$$H^{2i}(\overline{X(v)}) = \bigoplus_{\substack{z<v\\ \ell(z)=i}} H(v)_z$$

(here the $H(v)_z$ do not necessarily coincide with the expressions in (3.3) such that the complex

$$(3.4)\quad H^{2i}(\overline{X(w)}) \longrightarrow \bigoplus_{\substack{u\leq v\leq w\\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X(v)}) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{u\leq v\leq w\\ \ell(v)=\ell(u)+1}} H^{2i}(\overline{X(v)}) \longrightarrow H^{2i}(\overline{X(u)})$$

is quasi-isomorphic to the graded complex

$$(3.5)\quad \bigoplus_{\substack{z\leq w\\ \ell(z)=i}} H(w)_z \longrightarrow \bigoplus_{\substack{u\leq v\leq w\\ \ell(v)=\ell(w)-1}} \bigoplus_{\substack{z\leq v\\ \ell(z)=i}} H(v)_z \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{u\leq v\leq w\\ \ell(v)=1}} \bigoplus_{\substack{z\leq v\\ \ell(z)=i}} H(v)_z \longrightarrow \bigoplus_{\substack{z\leq u\\ \ell(z)=i}} H(u)_z$$

$$= \bigoplus_{\substack{z\leq w\\ \ell(z)=i}} \Big(H(w)_z \longrightarrow \bigoplus_{\substack{u\leq v\leq w\\ \ell(v)=\ell(w)-1}} H(v)_z \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{u\leq v\leq w\\ \ell(v)=1}} H(v)_z \longrightarrow \bigoplus_{\substack{z\leq u\\ \ell(z)=i}} H(u)_z \Big).$$

We shall prove a more concrete version of the conjecture in the case where $\ell(u) = \ell(w)-1$ and $i = 1$.

Lemma 3.11. *Let w be a Coxeter element and let $w' \in W$ with $w' \leq w$ and $\ell(w') = \ell(w)-1$. There are gradings $H^2(\overline{X(w)}) = \bigoplus_{\substack{z\leq w \\ \ell(z)=1}} H(w)_z$ and $H^2(\overline{X(w')}) = \bigoplus_{\substack{z\leq w' \\ \ell(z)=1}} H(w')_z$ such that the induced homomorphism $H^2(\overline{X(w)}) \longrightarrow H^2(\overline{X(w')})$ is quasi-isomorphic to the graded one. Moreover, the maps $H(w)_z \longrightarrow H(w')_z$ are injective for all $z \leq w'$.*

Proof. If $w' = s_2 \cdots s_h \in W$, the claim is a result of the proof of Proposition 3.4 following inductively the process of blow ups. Here we may consider on $H^2(\overline{X(w)})$ and $H^2(\overline{X(w')}) = \mathrm{Ind}^G_{P_{(1,n-1)}}\big(H^2(\overline{X_{\mathrm{GL}_1\times \mathrm{GL}_{n-1}}(w')})\big)$ the natural gradings given by loc.cit., i.e.

$$H(w)_{s_1} = i^G_G(-1) \text{ and } H(w)_{s_{i+1}} = i^G_{P_{(i,n-i)}}(-1),\ i\geq 1,$$

resp.

$$H(w')_{s_2} = i^G_{P_{(1,n-1)}}(-1) \text{ and } H(w')_{s_{i+1}} = i^G_{P_{(1,i-1,n-i)}}(-1),\ i\geq 2.$$

If on the other extreme $w' = s_1 \cdots s_{h-1}$, then we identify $\overline{X(\mathrm{Cox}_n)}$ with $\overline{X(s_h s_{h-1}\cdots s_2 s_1)}$ (cf. Remark 3.5 i)) and argue in the same way as above using the variant of Proposition 3.4 (cf. Remark 3.5 iii)) describing the latter space as a sequence of blow ups using hyperplanes. Here the gradings are induced by mirroring the Dynkin diagram and the induced representations, i.e. by setting

$$H(w)_{s_h} = i^G_G(-1) \text{ and } H(w)_{s_i} = i^G_{P_{(i+1,n-i-1)}}(-1),\ i<h,$$

resp.

$$H(w')_{s_{h-1}} = i^G_{P_{(n-1,1)}}(-1) \text{ and } H(w')_{s_i} = i^G_{P_{(i+1,n-i-2,1)}}(-1),\ i<h-1.$$

In general, i.e. if $w' = s_1 \cdots \hat{s_i} \cdots s_h \in W$ for some $1 < i < h-2$, then the recipe is a mixture of both approaches. By Remark 3.5, the cycle map $A^2(\overline{X(w)})_{\overline{\mathbb{Q}}_\ell} \longrightarrow H^2(\overline{X(w)})$ is an isomorphism. The same holds true for the subvariety $\overline{X(w')}$. We set

$$H(w)_{s_j} = \begin{cases} i^G_{P_{(j-1,n-j+1)}}(-1) & ;\ j>i \\ i^G_G(-1) & ;\ j=i \\ i^G_{P_{(i-j,n-i+j)}}(-1) & ;\ j<i \end{cases}$$

Here the contribution $H(w)_z = i^G_{P_{(k,n-k)}}(-1)$ is induced by the cycles $\{V^\bullet \in \overline{X}(w) \mid V^k$ is rational $\}$. Further we set

$$H(w')_{s_j} = \begin{cases} i^G_{P_{(i,j-(i+1),n-j+1)}}(-1) & ;\ j > i+1 \\ \\ i^G_{P_{(i,n-i)}}(-1) & ;\ j = i+1, j = i-1 \\ \\ i^G_{P_{(j+1,i-j-1,n-i)}}(-1) & ;\ j < i-1 \end{cases} .$$

Here the contributions are induced by the corresponding cycles. The statement follows from the transversality of each of the subvarieties $\overline{X(v)}$, $v \neq w'$, with $\overline{X(w')}$. Furthermore, the self intersection of $\overline{X}(w')$ gives rise to the contribution $H(w)_{s_{i+1}} \longrightarrow H(w')_{s_{i+1}}$. The claim follows. $\square$

Remark 3.12. If $w' = s_2 \cdots s_h \in W$ or $w' = s_1 \cdots s_{h-1} \in W$ it follows again directly from Proposition 3.4 that the corresponding claim is true for all cohomology degrees.

To the end of this section we recall the definition of Deligne-Lusztig varieties attached to elements of the Braid monoid B^+ of W and to the description of smooth compactifications of them, cf. [DMR], [DL]. The Braid monoid B^+ is the quotient of F^+ given by the relations $(st)^{m_{s,t}} = 1$ where $s,t \in S$ with $s \neq t$. Here $m_{s,t} \in \mathbb{Z}$ is the order of the element $st \in W$. Thus we have surjections

$$F^+ \xrightarrow{\alpha} B^+ \xrightarrow{\beta} W$$

with $\gamma = \beta \circ \alpha$. There is a section $W \hookrightarrow B^+$ of β which identifies W with the subset

$$B^+_{\mathrm{red}} = \{w \in B^+ \mid \ell(w) = \ell(\beta(w))\}$$

of reduced elements in B^+, cf. [GKP]. In the sequel we will identify W with B^+_{red}.

For any element $w = s_{i_1} \cdots s_{i_r} \in F^+$, set

$$\begin{aligned} X(w) &:= X(s_{i_1}, \ldots, s_{i_r}) \\ &:= \Big\{x = (x_0, \ldots, x_r) \in X^{r+1} \mid \mathrm{inv}(x_{j-1}, x_j) = s_{i_j}, j = 1, \ldots, r,\ x_r = F(x_0)\Big\}. \end{aligned}$$

This is a smooth variety over k equipped with an action of G. If $w \in W$ and $w = s_{i_1} \cdots s_{i_r}$ is a fixed reduced decomposition, then we also simply write $X^{F^+}(w)$ for $X(s_{i_1}, \ldots, s_{i_r})$. For any $w \in W$, the map

$$\begin{aligned} X(s_{i_1}, \ldots, s_{i_r}) &\longrightarrow X(w) \\ (x_0, \ldots, x_r) &\mapsto x_0 \end{aligned} \tag{3.6}$$

defines a G-equivariant isomorphism of varieties over k. Moreover by Broué, Michel [BM] and Deligne [De] the variety $X^{F^+}(w)$ depends up to an unique isomorphism only on the image of $s_{i_1}\cdots s_{i_r}$ in B^+. Finally, we set for $w=s_{i_1}\cdots s_{i_r}\in F^+$,

$$\begin{aligned}\overline{X}(w) &:= \overline{X}(s_{i_1},\ldots,s_{i_r})\\ &:= \Big\{x=(x_0,\ldots,x_r)\in X^{r+1}\mid \mathrm{inv}(x_{j-1},x_j)\in\{e,s_{i_j}\}, j=1,\ldots,r, x_r=F(x_0)\Big\}.\end{aligned}$$

Again this is a k-variety with an action of G which includes $X(w)$ as an open subset. More precisely, for all $v\preceq w$ we can identify $X(v)$ with a locally closed subvariety of $\overline{X}(w)$ and we get in this way a stratification $\overline{X}(w)=\bigcup_{v\preceq w}X(v)$.

Proposition 3.13. *For all $w\in F^+$, the variety $\overline{X}(w)$ is smooth and projective.*

Proof. See [DL, 9.10] resp. [DMR, Prop. 2.3.5, 2.3.6]. □

Hence if $w\in W$ and $w=s_{i_1}\cdots s_{i_r}$ is a reduced decomposition, then the variety

$$\overline{X}^{F^+}(w):=\overline{X}(s_{i_1},\ldots,s_{i_r})$$

is a smooth compactification of $X(w)$. The map (3.6) extends to a surjective proper birational morphism

$$\pi:\overline{X}^{F^+}(w)\longrightarrow\overline{X(w)}$$

which we call the Bott-Samelson or Demazure resolution of $X(w)$ with respect to the reduced decomposition.

Remark 3.14. If w is a Coxeter element, then the map π is an isomorphism. In fact, this follows easily by considering the natural stratifications on both sides.

Remark 3.15. When $w\in F^+$ is not full, then the obvious variant of Proposition 3.6 does also hold true for the DL-varieties $X(w)$ and their compactifications $\overline{X}(w)$, cf. [DMR, Cor. 3.1.3].

4. Squares

We consider the action of W on itself by conjugation. The set of conjugacy classes C in W is in bijection with the set of partitions $\mathcal{P}$ of n. For a partition $\mu\in\mathcal{P}$, let C^μ be the corresponding conjugacy class. Let $C_{\min}$ be the set of minimal elements in a given conjugacy class C. For $\mu=(\mu_1,\ldots,\mu_r)\in\mathcal{P}$, let $\mu'=(\mu'_1,\ldots,\mu'_s)\in\mathcal{P}$ be the conjugate partition. Then the set $C^\mu_{\min}$ consists precisely of the Coxeter elements in $W_\mu:=\prod_{i=1}^s S_{\mu'_i}$. All elements in $C_{\min}$ have the same length.

We recall the following result of Geck, Kim and Pfeiffer. Let $w, w' \in W$ and $s \in S$. Set $w \xrightarrow{s} w'$ if $w' = sws$ and $\ell(w') \leq \ell(w)$. We write $w \rightarrow w'$ if $w = w'$ or if there are elements $s_1, \ldots, s_r \in S$ and $w = w_1, \ldots, w_r = w' \in W$ with $w_i \xrightarrow{s_i} w_{i+1}$, $i = 1, \ldots, r-1$.

Theorem 4.1. ((([GKP], Thm. 2.6) *Let C be a conjugacy class of W. For any $w \in C$, there exists some $w' \in C_{min}$ such that $w \rightarrow w'$.* □

As a consequence, for any $w \in W$, there exists a finite set of cyclic shifts (i.e. elementary conjugations $w \longrightarrow sws$, $s \in S$) such that the resulting element has the shape $sw's$ with $\ell(w) = \ell(w') + 2$.

Let $s \in S$ and let $w, w' \in W$ with $w = sw's$. Suppose that $\ell(w) = \ell(w') + 2$. We put

$$Z = X(w) \cup X(sw') \text{ and } Z' = X(w's) \cup X(w')$$

and

$$\tilde{Z} = X(w) \cup X(w's) \text{ and } \tilde{Z}' = X(sw') \cup X(w').$$

Proposition 4.2. *(Deligne-Lusztig) i) The varieties $X(w's)$ and $X(sw')$ are universally homeomorphic by maps $\sigma : X(sw') \longrightarrow X(w's)$ and $\tau : X(w's) \longrightarrow X(sw')$ with $\tau \circ \sigma = F$ and $\sigma \circ \tau = F$. Hence $H^*_c(X(w's)) \cong H^*_c(X(sw'))$.*

ii) The above maps extend to morphisms $\tau : Z' \longrightarrow \tilde{Z}'$ and $\sigma : \tilde{Z}' \longrightarrow Z'$ with $\sigma|X(w') = id$ and $\tau|X(w') = F$.

Proof. i) For later use we just recall the construction of the maps and refer for the proofs to [DL]. Let $B \in X(sw')$. Then there is by property (1) b) of the varieties $O(-)$ a unique Borel subgroup $\sigma(B) \in X$ with $(B, \sigma(B)) \in \emptyset(s)$ and $(\sigma B, F(B)) \in \emptyset(w')$. A quick computation shows that $(\sigma(B), F(\sigma B)) \in \emptyset(w's)$, cf. [DL, Thm 1.6]. The map τ is defined analogously.

ii) This follows easily from the definitions of the maps σ, τ. □

Corollary 4.3. *Let C be a conjugacy class and let $v, w \in C_{\min}$. Then $H^*_c(X(v)) \cong H^*_c(X(w))$.*

Proof. By using the previous proposition, it is easily verified that the statement is true for all Coxeter elements in G. On the other hand, minimal elements in a conjugacy class are precisely the Coxeter elements in the corresponding Levi subgroup. Moreover, the Levi subgroups to v and w are associate, hence we deduce the claim by Proposition 3.6. □

The following statements seem to be known to the experts (e.g. confer the proof of [DMR, Prop. 3.2.10]).

Proposition 4.4. *There is a $\mathbb{A}^1$-bundle $\gamma : Z \longrightarrow Z'$.*

Proof. As in the proof of Theorem 1.6 in [DL], we may write $X(w)$ as a (set-theoretical) disjoint union

$$X(w) = X_1 \cup X_2$$

where X_1 is closed in $X(w)$ and X_2 is its open complement (Note that we have interchanged the role of w and w' compared to [DL]). Recall the definition of $X_i, i = 1, 2$. For $B \in X(w)$ there are unique Borel subgroups $\delta(B), \gamma(B) \in X$ with $(B, \gamma(B)) \in \O(s)$, $(\gamma(B), \delta(B)) \in \O(w')$ and $(\delta(B), F(B)) \in \O(s)$. Then

$$X_1 = \big\{B \in X(w) \mid \delta(B) = F(\gamma(B))\big\} \text{ resp. } X_2 = \big\{B \in X(w) \mid \delta(B) \neq F(\gamma(B))\big\}.$$

In [DL] it is shown that the map $\gamma : X_1 \longrightarrow X(w')$ is a $\mathbb{A}^1$-bundle whereas $\delta : X_2 \longrightarrow X(w's)$ is a $\mathbb{G}_m$-bundle.

On the other hand, if $B \in X_2$, then $(\delta(B), F(B)) \in \O(s)$, $(F(B), F(\gamma B)) \in \O(s), \delta(B) \neq F(\gamma B)$, hence $(\delta(B), F(\gamma(B))) \in \O(s)$. This was already shown in [DL]. It follows that $(\gamma(B), F(\gamma(B))) \in \O(w's)$, hence $\gamma(B) \in X(w's)$. Thus we have a morphism $\gamma : X(w) \longrightarrow Z'$ of varieties compatible with the action of G and F.

By Proposition 4.2 there is a homeomorphism $\sigma : X(sw') \longrightarrow X(w's)$. This map is compatible with $\gamma : X(w) \longrightarrow Z'$ in the sense that both maps glue to a morphism $\gamma : Z \longrightarrow Z'$. This map is clearly an $\mathbb{A}^1$-bundle. □

Corollary 4.5. *There is an isomorphism $H^i_c(Z) = H^{i-2}_c(Z')(-1)$ for all $i \geq 2$.* □

The map γ even extends to a larger locally closed subvariety as follows. We set $\hat{Z} := Z \cup Z'$.

Lemma 4.6. *The set $\hat{Z}$ is locally closed in X.*

Proof. For proving the assertion, it suffices to show (by considering topological closures of DL-varieties) that there is no element $v \in W$ different from sw' resp. $w's$ with $w' \leq v \leq w$ This proof is a consequence of the following result. □

Lemma 4.7. *Let $w_1, w_2 \in W$ with $w_1 \leq w_2$ and $\ell(w_2) = \ell(w_1) + 2$. Then there are uniquely determined elements $v_1, v_2 \in W$ with $w_1 \leq v_i \leq w_2$ and $\ell(v_i) = \ell(w_1) + 1$.*

Proof. See [BGG, Lemma 10.3]. □

In the above situation, Bernstein, Gelfand, Gelfand call the quadruple $Q = \{w_1, v_1, v_2, w_2\}$ a square in W. Here we use sometimes the graphical illustration of [Ku] (resp. for technical reasons sometimes without arrows) to indicate this kind of object:

$$Q: \quad \begin{tikzcd} & w_2 & \\ v_1 \arrow[ur] & & v_2 \arrow[ul] \\ & w_1 \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \left(\text{ resp. } \begin{array}{ccc} & w_2 & \\ v_1 & & v_2 \\ & w_1 & \end{array} \right).$$

Lemma 4.6 generalizes as follows.

Lemma 4.8. *For any square $Q = \{w_1, v_1, v_2, w_2\}$ in W, the subset $X(Q) := X(w_2) \cup X(v_2) \cup X(v_1) \cup X(w_1)$ is locally closed in X.* □

For later use, we also mention the given-below property.

Lemma 4.9. *For any square $Q = \{w_1, v_1, v_2, w_2\} \subset W$, let*

$$\delta^{i-1}_{w_1, v_1 \cup v_2} : H^{i-1}_c(X(w_1)) \longrightarrow H^i_c(X(v_1) \cup X(v_2))$$

and

$$\delta^{i}_{v_1 \cup v_2, w_2} : H^{i}_c(X(v_1) \cup X(v_2)) \longrightarrow H^{i+1}_c(X(w_2))$$

be the corresponding boundary homomorphism. Then $\delta^{i}_{v_1 \cup v_2, w_2} \circ \delta^{i-1}_{w_1, v_1 \cup v_2} = 0$.

Proof. This is clear as the map $\delta^{i}_{v_1 \cup v_2, w_2} \circ \delta^{i-1}_{w_1, v_1 \cup v_2}$ is just the composite of the corresponding differentials in the E_1-term associated to the stratification $X(Q) = X(w_2) \,\dot{\bigcup}\, \big(X(v_1) \cup X(v_2)\big) \,\dot{\bigcup}\, X(w_1)$. □

Now we come back to the the locally closed subvariety $\hat{Z} \subset X$.

Proposition 4.10. *The map γ extends to a $\mathbb{P}^1$-bundle $\hat{Z} \longrightarrow Z'$ with $\gamma|_{Z'} = id_{Z'}$.*

Proof. This is a direct consequence of the definitions of γ and the variety Z' realising the latter space as the set $\{(B_0, B_1, B_2, B_3) \in X^4 \mid (B_0, B_1) \in \emptyset(e), (B_1, B_2) \in \emptyset(w'), (B_2, B_3) \in \overline{\emptyset(s)}, B_3 = F(B_0)\}$. □

Corollary 4.11. *There is an isomorphism of H-modules*

$$H^i_c(\hat{Z}) = H^i_c(Z') \oplus H^{i-2}_c(Z')(-1)$$

for all $i \geq 0$. □

For the next statement, we consider the open subset $Y := X(w) \cup X(sw') \cup X(w's)$ of $\hat{Z}$.

Corollary 4.12. *There is a natural splitting $H^i_c(Y) = H^i_c(Z) \oplus H^i_c(X(w's))$ as H-modules for all $i \geq 0$.*

Proof. The existence of a splitting is easily verified by considering the diagram of long exact cohomology sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^i_c(Z) & \longrightarrow & H^i_c(\hat{Z}) & \longrightarrow & H^i_c(Z') & \xrightarrow{\delta^i} & H^{i+1}_c(Z) & \longrightarrow \cdots \\
 & \| & & \uparrow & & \uparrow & & \| & \\
\cdots \longrightarrow & H^i_c(Z) & \longrightarrow & H^i_c(Y) & \longrightarrow & H^i_c(X(w's)) & \longrightarrow & H^{i+1}_c(Z) & \longrightarrow \cdots
\end{array}$$

together with the fact that the differential map δ^i vanishes. That it is natural comes about from the fact that the subset $U := X_2(w) \cup X(sw') \cup X(w's)$ is open in Y and we have a $\mathbb{P}^1$-bundle $\gamma : U \longrightarrow X(w's)$ with $\gamma \circ i = \mathrm{id}$ where $i : X(w's) \hookrightarrow U$ is the inclusion. □

In the sequel, we denote by

$$r^i_{w,sw'} : H^i_c(Z) \longrightarrow H^i_c(X(sw'))$$

the map which is induced by the closed immersion $X(sw') \hookrightarrow Z$. We consider the corresponding long exact cohomology sequence

$$\cdots \longrightarrow H^{i-1}_c(X(sw')) \longrightarrow H^i_c(X(w)) \longrightarrow H^i_c(Z) \longrightarrow H^i_c(X(sw')) \longrightarrow \cdots$$

which by Corollary 4.5 identifies with the sequence

$$\begin{aligned}
\cdots &\longrightarrow H^{i-1}_c(X(sw')) \longrightarrow H^i_c(X(w)) \longrightarrow H^{i-2}_c(X(w's) \cup X(w'))(-1) \\
&\longrightarrow H^i_c(X(sw')) \longrightarrow \cdots .
\end{aligned} \tag{4.1}$$

Remark 4.13. In [DMR] it is proved that there is a long exact cohomology sequence

$$\begin{aligned}
\cdots \longrightarrow \quad & H^i_c(X(w)) \longrightarrow H^{i-2}_c(X(w'))(-1) \longrightarrow H^{i-1}_c(X(sw'))(-1) \oplus H^i_c(X(sw')) \\
\longrightarrow \quad & H^{i+1}_c(X(w)) \longrightarrow \cdots
\end{aligned}$$

which relies on the fact that the induced maps $\delta^i : H^{i-2}_c(X(sw'))(-1) \longrightarrow H^i_c(X(sw'))$ induced by the $\mathbb{G}_m$-bundle X_2 over $X(sw')$ is trivial (as already stated in [DL, Thm. 1.6]). In particular, it follows that the cokernel of the boundary map $H^{i-3}_c(X(w')) \longrightarrow H^{i-2}_c(X(w's))$ always contributes to $H^i_c(X(w))$.

Remark 4.14. The same statements presented here (Prop. 4.4. - Cor. 4.12) are true if we work with elements in F^+ instead in W, cf. also [DMR]. More precisely, if $w = sw's$ for $w, w' \in F^+$ and $s \in S$, then we can define subsets $X_1, X_2 \subset X(w)$ such that X_1 is a $\mathbb{A}^1$-bundle over $X(w')$ and such that X_2 is an $\mathbb{G}_m$-bundle over $X(w's)$. With the same reasoning, the subset $X(w) \cup X(sw')$ is an $\mathbb{A}^1$-bundle over $X(w's) \cup X(w')$, etc.

Example 4.15. We reconsider Example 3.1 (which is also discussed in [DMR, ch. 4]). So let $w_0 = sw's = (1,3)$ with $s = s_2 = (2,3), w' = s_1 = (1,2)$. We are going to determine the cohomology of the DL-variety $X(w_0)$. The cohomology of $X(s_1s_2)$ resp. $X(s_2s_1)$ is given by Proposition 3.3 by

$$H_c^*(X(s_2s_1)) = H_c^*(X(s_1s_2)) = v_B^G[-2] \oplus v_{P_{(2,1)}}^G(-1)[-3] \oplus i_G^G(-2)[-4].$$

Furthermore we have $H_c^*(X(w')) = i_B^G/i_{P_{(2,1)}}^G[-1] \oplus i_{P_{(2,1)}}^G(-1)[-2]$. Now the variety $Z' = X(w's) \cup X(w')$ coincides with the set $\{V^\bullet \mid F(V^1) \subset V^2, V^1 \neq F(V^1)\}$ which we may identify with $\mathbb{P}(V) \setminus \mathbb{P}(V)(k)$. Hence we obtain (which follows also by applying Proposition 5.8)

$$H_c^*(Z') = v_{P_{(2,1)}}^G[-1] \oplus i_G^G(-1)[-2] \oplus i_G^G(-2)[-4]$$

and therefore

$$H_c^*(Z) = v_{P_{(2,1)}}^G(-1)[-3] \oplus i_G^G(-2)[-4] \oplus i_G^G(-3)[-6]$$

by Corollary 4.5. We claim that the maps $r_{w,sw'}^3, r_{w,sw'}^4$ are surjective. Indeed for $i = 4$ this is clear since $H_c^4(X(sw'))$ is the top cohomology group of $X(sw')$. As for $i = 3$ we consider the natural map

$$H_c^3(Y) \longrightarrow H_c^3(X(sw')) \oplus H_c^3(X(w's)).$$

Applying Corollary 4.12 we get a splitting $H_c^3(Y) = H_c^3(Z) \oplus H_c^3(X(w's))$ such the the second summand maps onto $H_c^3(X(w's))$ via the identity map and onto $H_c^3(X(sw'))$ by σ^* where $\sigma : X(sw') \longrightarrow X(w's)$ is the homeomorphism of Proposition 4.2. Now we rewrite w_0 in the shape $w_0 = s'w''s'$ with $s' = (1,2)$ and $w'' = (2,3)$. Applying Corollary 4.12 to this data we get another splitting $H_c^3(Y) = H_c^3(\tilde{Z}) \oplus H_c^3(X(sw'))$ as $w''s' = sw'$. Thus the second summand maps to $H_c^3(X(sw'))$ via the identity and onto $H_c^3(X(w's))$ via τ^*. As the Tate twist is non-trivial we deduce that $H_c^3(X(sw')) \neq H_c^3(X(w's))$ in $H_c^3(Y)$. It follows that the map $r_{w,sw'}^3 : H_c^3(Z) \longrightarrow H_c^3(X(sw'))$ is surjective. Hence we get

$$H_c^*(X(w_0)) = v_B^G[-3] \oplus i_G^G(-3)[-6].$$

5. Cohomology of DL-varieties of height one

In this section we determine the cohomology of DL-varieties attached to Weyl group elements which are slightly larger than Coxeter elements, i.e. to elements which are of height one. For the definition of the height function we recall that by Theorem 4.1 there is for any $w \in W$ some element $w' \in W$ with $\ell(w) = \ell(w') + 2$ and $w \longrightarrow w'$.

Definition 5.1. We define the height of w inductively by $\mathrm{ht}(w) = \mathrm{ht}(w') + 1$. Here we set $\mathrm{ht}(w) = 0$ if w is minimal in its conjugacy class.

The proof of the next statement is immediate.

Lemma 5.2. *Let $w_{\min} \in W$ be a minimal element lying in the conjugacy class of w. Then $\ell(w) = \ell(w_{\min}) + 2\,\mathrm{ht}(w)$.* □

For an irreducible H-representation $V = j_\mu(-i)$, $\mu \in \mathcal{P}$, we set $t(V) = i \in \mathbb{N}$.

Proposition 5.3. *Let $v, w \in W$, $i, j, m \in \mathbb{Z}_{\geq 0}$ and suppose that $\mathrm{ht}(v) = 0$. Let $V \subset H^i_c(X(w))$ be a subrepresentation such that $V(m) \subset H^j_c(X(v))$. Then*

$$\ell(w) - \ell(v) + m \geq i - j \geq \ell(w) - \ell(v) + m - \mathrm{ht}(w).$$

Proof. As $\mathrm{ht}(v) = 0$ we deduce by Proposition 3.3 and Proposition 3.6 that $j = \ell(v) + t(V(m))$. Since $m = t(V) - t(V(m))$ we may assume that $v = 1$ and therefore $j = 0$.

We start with the case where $\mathrm{ht}(w) = 0$. In this case one has even - by looking again at Proposition 3.3 - the stronger identity

$$i = \ell(w) + m.$$

Now let $h(w) \geq 1$ and suppose that $w = sw's$. Consider the long exact cohomology sequence (4.1). We distinguish the following cases:

Case a) Let $V \subset H^{i-1}_c(X(sw'))$. By induction on the length we deduce that $\ell(sw') + m \geq i - 1 \geq \ell(sw') + m - \mathrm{ht}(sw')$. As $\ell(sw') = \ell(w) - 1$ and $\mathrm{ht}(w) \geq \mathrm{ht}(sw')$, we see that $\ell(w) + m \geq i \geq \ell(w) + m - h(w)$.

Case b) Let $V \subset H^i_c(Z)$. Then we must have $m \geq 1$.

Subcase i) Let $V(1) \subset H^{i-2}_c(X(sw'))$. By induction on the length we deduce that $\ell(sw') + m - 1 \geq i - 2 \geq \ell(sw') + m - 1 - h(sw')$. As $\ell(sw') = \ell(w) - 1$ and $\mathrm{ht}(w) \geq \mathrm{ht}(sw')$, we see that $\ell(w) + m \geq i \geq \ell(w) + m - \mathrm{ht}(w)$.

Subcase ii) Let $V(1) \subset H^{i-2}_c(X(w'))$. By induction on the length we deduce that $\ell(w') + m - 1 \geq i - 2 \geq \ell(w') + m - 1 - \mathrm{ht}(w')$. As $\ell(w') = \ell(w) - 2$ and $h(w) = h(w') + 1$, we see that even the stronger identity $\ell(w) + m - 1 \geq i \geq \ell(w) + m - h(w)$ holds true. □

Corollary 5.4. *Let $v, w \in W$ and $i, j, m \in \mathbb{Z}_{\geq 0}$. Let $V \subset H^i_c(X(w))$ be a subrepresentation such that $V(m) \subset H^j_c(X(v))$. Then*

$$\ell(w) - \ell(v) + m + \mathrm{ht}(v) \geq i - j \geq \ell(w) - \ell(v) + m - \mathrm{ht}(w).$$

Proof. We apply the foregoing proposition where w is replaced by v and v by 1. Then $\ell(v) + t(V(m)) \geq j \geq \ell(v) + t(V(m)) - \mathrm{ht}(v)$. Multiplying this term with -1 and adding the result to the sequence of inequalities $\ell(w) + t(V) \geq i \geq \ell(w) + t(V) - \mathrm{ht}(w)$ gives the statement. □

Definition 5.5. Let $w \in W$ and let $\mu \in \mathcal{P}$ be the partition such that $w \in C^\mu$. Then $\mu := \mu(w)$ is called the type of w.

Example 5.6. Let $w \in W$ be a Coxeter element, then $\mu(w) = (1, \ldots, 1)$. On the other extreme $\mu(1) = (n)$.

We also mention the following property which is not needed in the sequel.

Lemma 5.7. *Let $w = sw's \in W$ with $s \in S$ and $\ell(w) = \ell(w') + 2$.*

i) If $\mathrm{ht}(w) = \mathrm{ht}(sw') = \mathrm{ht}(w's)$ then $\mu(w) < \mu(sw')$.

ii) If $\mathrm{ht}(w) = \mathrm{ht}(sw') + 1$ then $\mu(w) > \mu(sw')$. □

For $w' \in W$ with $w' \leq w$ and $\ell(w') = \ell(w) - 1$, the set $Z' = X(w) \cup X(w')$ is a subvariety of X as already observed above. Here $X(w')$ is closed and $X(w)$ is open in Z'. We denote by

$$\delta^*_{w',w} : H^*_c(X(w')) \longrightarrow H^{*+1}_c(X(w))$$

the associated boundary map.

Proposition 5.8. *Let w be a Coxeter element and let $w' \in W$ with $w' \leq w$ and $\ell(w') = \ell(w) - 1$. Then the boundary homomorphism $\delta^j_{w',w} : H^j_c(X(w')) \longrightarrow H^{j+1}_c(X(w))$ is surjective for all $j \leq 2\ell(w') = 2(n-2)$. (In particular, $\delta^j_{w',w}$ is si-surjective for all $j = 0, \ldots, 2\ell(w) = 2(n-1)$.)*

Proof. We may suppose that $w = \mathrm{Cox}_n$. Since all the representations $H^i_c(X(w)) \neq (0)$ are irreducible, it suffices to show that the boundary maps $\delta^i_{w',w}$ for $i < 2\ell(w) - 1$, are non-trivial. Let $w' = s_1 \cdots \hat{s}_i \cdots s_h$ be as above. In terms of flags the DL-varieties in question have the following description

$$X(w) = \{V^\bullet \mid F(V^j) \subset V^{j+1}, F(V^j) \neq V^j,\ 1 \leq j \leq n-1\},$$

$$X(w') = \{V^\bullet \mid F(V^j) \subset V^{j+1}, F(V^i) = V^i, F(V^j) \neq V^j,\ 1 \leq j \neq i \leq n-1\}.$$

Their Zariski closures are given by

$$\overline{X(w)} = \{V^\bullet \mid F(V^j) \subset V^{j+1},\ 1 \leq j \leq n-1\},$$

$$\overline{X(w')} = \{V^\bullet \mid F(V^j) \subset V^{j+1}, F(V^i) = V^i,\ 1 \leq j \neq i \leq n-1\}.$$

The complement of $X(w)$ in $\overline{X(w)}$ is a divisor $D = \bigcup_W D_W$ where the union is over all k-rational subspaces W of V. For any rational flag $W^\bullet = (0) \subsetneq W^{i_1} \subsetneq W^{i_2} \subsetneq \cdots \subsetneq W^{i_k} \subsetneq V$

of V, we set $D_{W^\bullet} = D_{W^{i_1}} \cap D_{W^{i_2}} \cap \cdots \cap D_{W^{i_k}}$ and $\lg(W^\bullet) = k$. This construction gives rise for any constant sheaf A on $\overline{X(w)}$ to a resolution

$$A \longrightarrow \bigoplus_W A_{D_W} \longrightarrow \bigoplus_{W^\bullet, \lg(W^\bullet)=2} A_{D_{W^\bullet}} \longrightarrow \cdots \longrightarrow \bigoplus_{W^\bullet, \lg(W^\bullet)=n-1} A_{D_{W^\bullet}}.$$

of $A_{X(w)}$. On the other hand, we have $\overline{X(w')} = \bigcup_{W\in \mathrm{Gr}_i(V)(k)} D_W$. Similarly as above, we get a resolution

$$A_{\overline{X(w')}} \longrightarrow \bigoplus_{\substack{W^\bullet, \lg(W^\bullet)=2 \\ W^i\in W^\bullet}} A_{D_{W^\bullet}} \longrightarrow \bigoplus_{\substack{W^\bullet, \lg(W^\bullet)=3 \\ W^i\in W^\bullet}} A_{D_{W^\bullet}} \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{W^\bullet, \lg(W^\bullet)=n-1 \\ W^i\in W^\bullet}} A_{D_{W^\bullet}}$$

of $A_{X(w')}$. The second complex is a subcomplex of the first one and this inclusion induces the boundary map. In other terms, applying $H^{2i}(-)$ to both resolutions, we just get the complexes

$$H^{2i}(\overline{X(w)}) \longrightarrow \bigoplus_{\substack{v<w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X(v)}) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{v<w \\ \ell(v)=1}} H^{2i}(\overline{X(v)}) \longrightarrow H^{2i}(\overline{X(e)})$$

and

$$H^{2i}(\overline{X(w')}) \longrightarrow \bigoplus_{\substack{v<w' \\ \ell(v)=\ell(w')-1}} H^{2i}(\overline{X(v)}) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{v<w' \\ \ell(v)=1}} H^{2i}(\overline{X(v)}) \longrightarrow H^{2i}(\overline{X(e)}).$$

If $w' = s_1 s_2 \cdots s_{n-2}$, then $X(w') \cong \coprod_H \Omega(H)$ with H running through all rational hyperplanes in $V = \mathbb{F}^n$. Further we may identify $X(w)$ with $\Omega(V) \subset \mathbb{P}(V)$. Here the result is well-known in the setting of period domains. In fact, by considering also the varieties $\Omega(E)$ with E a rational subspace of V, we get a stratification of the projective space $\mathbb{P}(V)$. Then the result follows by weight reasons and the cohomology formula in Proposition 3.3 with respect to the varieties $\Omega(E)$. Alternatively, one might use the fundamental complex in [O]. By symmetry the same reasoning applies to $w' = s_2 s_3 \cdots s_{n-1}$.

In general we distinguish the cases whether $j = 2\ell(w')$ or $j < 2\ell(w')$. Suppose first that $j = 2\ell(w')$. Here we consider the order $\trianglelefteq$ on the set $\{z \in W \mid z \leq w, \ell(z) = \ell(w) - 1\}$ induced by the appearing of z in the sequence of blow ups $B_1, \ldots, B_{n-2} = \overline{X}(w)$ described in Prop. 3.4, i.e.

$$s_1 s_2 \cdots s_{n-2} \lhd s_2 s_3 \cdots s_{n-1} \lhd \cdots \lhd s_1 s_2 \cdots s_{n-3} s_{n-1}.$$

Considering long exact cohomology sequences with compact support together with the projection formula [Fu, Prop. 8.3.c)] applied to the maps $\overline{X}(w) \longrightarrow B_i, i = 1, \ldots, n-3$, we conclude by dimension reasons that for $v \lhd z$ the contribution $H(w)_v$ lies in the kernel of the restriction map $H^j(\overline{X}(w)) \longrightarrow H^j(\overline{X}(z)) = i^G_{P(z)}$. On the other hand, for $v \trianglerighteq z$ and $v \neq \mathrm{Cox}_{n-1}$ the map is induced by the double coset of $e \in W$ in $W_{I(v)} \backslash W / W_{I(z)}$ (if $v = z = \mathrm{Cox}_{n-1}$,

then it is the natural inclusion $i_G^G \hookrightarrow i_{P(z)}^G$). In fact, if $v = s_1 s_2 \cdots s_{n-3} s_{n-1}$, then one verifies that the intersection product within $\overline{X}(w)$ of each irreducible component in $\overline{X}(z)$ and every component of the cycle in $\overline{X}(v)$ giving rise to $H(w)_v$ is non trivial. For $v \lhd s_1 s_2 \cdots s_{n-3} s_{n-1}$, one applies again additionally the projection formula for proving this. All in all, it follows that the contribution $H^j(\overline{X}(w'))$ does not lie in the image of the map $H^j(\overline{X(w)}) \longrightarrow \bigoplus_{\substack{v<w \\ \ell(v)=\ell(w)-1}} H^j(\overline{X(v)})$ and hence the claim.

If $j < 2\ell(w')$ then we argue as follows. Let $v = s_1 s_2 \cdots s_{n-2}$ and $v' = \gcd(w', v) = s_1 \cdots \hat{s}_i \cdots s_{n-2}$. By induction on n the map $H_c^{j-1}(X(v')) \longrightarrow H_c^j(X(v))$ is surjective. On the other hand, by what we have observed above the map $H_c^j(X(v)) \longrightarrow H_c^{j+1}(X(w))$ is surjective, as well. Using Lemma 4.9 we deduce that the map $H_c^j(X(w')) \longrightarrow H_c^{j+1}(X(w))$ is non-trivial. $\square$

The next two statements give the cohomology of all Weyl group elements having full support and which are of height 1. Arbitrary elements of height one are handled by Proposition 3.6.

Proposition 5.9. *Let $w = sw's$ where $w' \in W$ is a Coxeter element in some Levi subgroup of a proper maximal parabolic subgroup in G . Then the maps $r^j_{w,sw'} : H_c^j(Z) \longrightarrow H_c^j(X(sw'))$ are all surjective for $j > \ell(sw') = h$. (In particular they are si-surjective for all $j \geq 0$.)*

Proof. We start with the observation that sw' and $w's$ are both Coxeter elements in W. We may suppose that $w' = s_1 \cdots \hat{s}_i \cdots s_{n-1}$ and $s = s_i$. It is clear that $v_B^G \notin \operatorname{supp} H_c^h(Z)$. So let $j > h$ and suppose that $r^j_{w,sw'}$ is not surjective. Then the irreducible module $H_c^j(X(sw'))$ maps injectively into $H_c^{j+1}(X(w))$ via the boundary homomorphism $\delta^j_{sw',w}$. First let $i < n-1$. Set

$$w'' := s_i s_1 s_2 \cdots \hat{s}_i \cdots s_{n-2} = s_i w' s_{n-1}.$$

This is a Coxeter element in the parabolic subgroup $W_{(n-1,1)}$ of W with $w'' \leq s_i w'$ and $\ell(w'') = \ell(w')$. Consider the square

$$\begin{tikzcd} & w & \\ s_i w' \arrow[ur] & & w'' s_i \arrow[ul] \\ & w'' . \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The boundary homomorphism $\delta^{j-1}_{w'',s_iw'} : H_c^{j-1}(X(w'')) \longrightarrow H_c^j(X(s_i w'))$ is si-surjective by Proposition 5.8. On the other hand, the boundary map $\delta^{j-1}_{w'',w''s_i} : H_c^{j-1}(X(w'')) \longrightarrow H_c^j(X(w'' s_i))$ vanishes as the map $r^j_{w''s_i,w''} : H_c^{j-1}(X(w''s_i) \cup X(w'')) \longrightarrow H_c^{j-1}(X(w''))$ is

(si-)surjective by induction on n. Indeed, both elements $w'', w''s_i$ are of shape above and in the Weyl group of GL_{n-1}. The start of induction is Example 4.15. By Lemma 4.9 the composite $\delta^j_{s_iw',w} \circ \delta^{j-1}_{w'',s_iw'}$ vanishes. The result follows in this special case.

If $i = n-1$, then we set $w'' := s_1w's_{n-1}$ and consider $w's_{n-1}$ instead of $s_{n-1}w'$ and $s_{n-1}w''$ instead of $w''s_{n-1}$. Then the same argument goes through. $\square$

By Proposition 5.8 we are able to give a formula for the cohomology of these height 1 elements. Here we could give the description of the induced representation $H^*_c(X(w')) = \mathrm{Ind}^G_{P_{(i,n-i)}}(H^*_c(X_{\mathbf{M}}(w'))$ by using Littlewood-Richardson coefficients, cf. [FH, §A] (Note that the structure or combinatoric of unipotent G- and W-representations is the same, cf. Remark 2.2). Instead we prefer to use the notation which is common in the Grothendieck group of G-representations. Hence if we write $V - W$ for two G-representations V, W, then we mean implicitly that W is a subrepresentation of V.

Corollary 5.10. *In the situation of the foregoing proposition, we have for $j \in \mathbb{N}$, with $\ell(w) < j < 2\ell(w) - 1$,*

$$H^j_c(X(w)) = \big(H^{j-2}_c(X(w')) - j_{(j+1-n,1,\ldots,1)}(n-j)\big)(-1) - j_{(j+2-n,1\ldots,1)}(n-j-1).$$

Moreover, we have $H^{\ell(w)}_c(X(w)) = v^G_B \bigoplus \big(v^G_{P_{\mu(w)}} - j_{(2,1\ldots,1)}\big)(-1)$, $H^{2\ell(w)-1}_c(X(w)) = 0$ and $H^{2\ell(w)}_c(X(w)) = i^G_G(-\ell(w))$.

Proof. By Proposition 5.9, we deduce that $H^j_c(X(w)) = \ker\big(H^j_c(Z) \longrightarrow H^j_c(X(sw'))$ for all $j > \ell(w) = n$. By Proposition 3.3 we have $H^j_c(X(sw')) = j_{(j+2-n,1\ldots,1)}(n-j-1)$. Further $H^j_c(Z) = H^{j-2}_c(X(w's) \cup X(w'))(-1)$ and the boundary map

$$H^{j-2}_c(X(w')) \longrightarrow H^{j-1}_c(X(w's)) = j_{(j+1-n,1,\ldots,1)}(n-j)$$

is surjective for $j - 2 \leq 2\ell(w') = 2\ell(w) - 4$ by Proposition 5.8. Hence we get the first identity in the statement.

If $j = \ell(w)$ then one verifies easily that

$$H^{j-2}_c(X(w')) - j_{(j+1-n,1,\ldots,1)}(n-j) = H^{\ell(w')}_c(X(w')) - v^G_B = v^G_{P_{\mu(w)}}.$$

In addition the Steinberg representation appears as a summand. It is induced via the boundary map by $H^{\ell(w)-1}_c(X(w's)) = H^{\ell(w's)}_c(X(w's)) = v^G_B$.

The remaining identities for $j = 2\ell(w) - 1, 2\ell(w)$ are easily verified in the same way. $\square$

Remark 5.11. Let $w \in W$ have full support. Then we always have $H^{\ell(w)}_c(X(w)) \supset v^G_B = j_{(1,\ldots,1)}$ and $H^{2\ell(w)}_c(X(w)) = v^G_G(-\ell(w)) = j_{(n)}(-\ell(w))$, cf. [L2, Prop. 1.22], [DMR, Prop.

3.3.14, 3.3.15]. More precisely, these are the only cohomology degrees where these extreme unipotent representations appear. Further $H^i_c(X(w))=0$ for all $i<\ell(w)$.

Example 5.12. Let $n=4$, $w=(1,2)(2,3)(3,4)(1,2)\in W$. Then

$$H^*_c(X(w))=v_B^G[-4]\oplus j_{(2,2)}(-2)[-5]\oplus i_G^G(-4)[-8].$$

Example 5.13. Let $n=4$, $w=(2,3)(1,2)(3,4)(2,3)\in W$. Then

$$\begin{aligned} H^*_c(X(w)) &= v_B^G[-4]\oplus j_{(2,2)}(-1)[-4]\oplus j_{(2,1,1)}(-2)[-5] \\ &\quad \oplus j_{(3,1)}(-2)[-5]\oplus j_{(2,2)}(-3)[-6]\oplus i_G^G(-4)[-8]. \end{aligned}$$

The remaining elements with full support and which are of height 1 are treated by the next result.

Proposition 5.14. *Let $w=sw's\in W$ with $\ell(w)=\ell(w')+2$ for some Coxeter element $w'\in W$ and $s\in S$. Then the map $r^j_{w,sw'}:H^j_c(Z)\longrightarrow H^j_c(X(sw'))$ vanishes for all $j\neq 2\ell(w)-2$ and is an isomorphism for $j=2\ell(w)-2$. Hence we have*

$$H^j_c(X(w))=H^{j-2}_c(X(w's)\cup X(w'))(-1)\oplus H^{j-1}_c(X(sw'))$$

for all $j\neq 2\ell(w)-1, 2\ell(w)-2$ and $H^{2\ell(w)-1}_c(X(w))=H^{2\ell(w)-2}_c(X(w))=0$.

Proof. By Prop. 5.9 the boundary map $H^{j-2}_c(X(w'))\longrightarrow H^{j-1}_c(X(w's))$ vanishes for all $j\in\mathbb{N}$ with $j-2\neq\ell(w')=n-1$. If $j-2=\ell(w')$, then it is an injection, since on the LHS we have the Steinberg representation v_B^G. Hence

$$\begin{aligned} H^j_c(X(sw's)\cup X(sw')) &\cong H^{j-2}_c(X(w's)\cup X(w'))(-1) \\ &= H^{j-2}_c(X(w's))(-1)\oplus H^{j-2}_c(X(w'))(-1) \end{aligned}$$

for all $j>\ell(w')+2=n+1$. By Remark 4.13 we know that $r^j_{w,sw'}$ applied to a contribution of $H^{j-2}_c(X(w's))(-1)$ vanishes. On the other hand, by comparing weights we see that the Tate twist of a contribution in $H^j_c(X(sw's)\cup X(sw'))$ induced by $H^{j-2}_c(X(w'))(-1)$ is different from the Tate twist of $H^j_c(X(sw')$, except for $j=2\ell(w)-2$. Here we have the trivial representation on both sides. The result follows. ∎

Example 5.15. Let $n=4$, $w=(2,3)(1,2)(2,3)(3,4)(2,3)\in W$. Then one verifies that

$$\begin{aligned} H^*_c(X(w)) &= v_B^G[-5]\oplus j_{(2,2)}(-2)[-6]\oplus j_{(2,1,1)}(-2)[-6] \\ &\quad \oplus j_{(3,1)}(-3)[-7]\oplus j_{(2,2)}(-3)[-7]\oplus i_G^G(-5)[-10]. \end{aligned}$$

For determining the cohomology of the longest element in the Weyl group of GL_4 we make use of the next statement.

Proposition 5.16. *Suppose that we may write $w = sw's$ in another presentation $w = s'w''s'$ with $s' \neq s$ and $w' = w''$. Then the maps $r^i_{w,sw'} : H^i_c(X(w) \cup X(sw')) \longrightarrow H^i_c(X(sw'))$ are surjective for all $i > \ell(sw')$.*

Proof. Since $s \neq s'$ it follows that $w's \neq w''s'$, hence $w''s' = sw'$ by Lemma 4.7. By Proposition 4.12 we get splittings $H^i_c(Y) = H^i_c(Z) \oplus H^i_c(X(sw'))$ and $H^i_c(Y) = H^i_c(\tilde{Z}) \oplus H^i_c(X(w's))$. Now the argument is similar to the one in Example 4.15. Indeed, via these identifications the map $i^* : H^i_c(Y) \longrightarrow H^i(X(sw'))$ induced by the closed immersion $i : X(sw') \hookrightarrow Y$ is the identity on $H^i_c(X(sw'))$ whereas the contribution $H^i_c(X(w's))$ maps to $H^i_c(X(sw'))$ via τ^*. Thus $H^i_c(X(sw')) \neq H^i_c(X(w's))$ in $H^i_c(Y)$ since the Tate twist is non-trivial. More concretely, we consider the diagram

$$
\begin{array}{ccccc}
H^i_c(X(w's)) & & & & H^i_c(X(w's)) \\
& \searrow & & \nearrow & \\
& & H^i_c(Y) & & \\
& \nearrow & & \searrow & \\
H^i_c(X(w's)) & & & & H^i_c(X(w's)),
\end{array}
$$

where the maps on the LHS are induced by the splittings. On the RHS the maps are induced by the closed embedding $X(w's) \cup X(sw') \hookrightarrow Y$. The induced map

$$H^i_c(X(w's)) \oplus H^i_c(X(sw')) \longrightarrow H^i_c(X(w's)) \oplus H^i_c(X(w's))$$

is given by $(x, y) \mapsto (x - \sigma^*(y), y - \tau^*(x))$. Since $\sigma \circ \tau = F = \tau \circ \sigma$ and the Tate twist is non-trivial this map is an isomorphism. Thus the surjectivity of $r^i_{w,sw'}$ follows now easily. $\square$

Example 5.17. Let $G = \mathrm{GL}_4$. Let $w = w_0 = (1,4)(2,3)$. Here we may write $w = (3,4)(1,3)(2,4)(3,4) = (1,2)(1,3)(2,4)(1,2)$. One deduces easily that

$$
\begin{aligned}
H^*_c(X((1,4)(2,3))) \quad = \quad & v^G_B[-6] \oplus j_{(2,1,1)}(-2)[-7] \oplus j_{(2,2)}(-3)[-8]^2 \\
& \oplus j_{(3,1)}(-4)[-9] \oplus i^G_G(-6)[-12].
\end{aligned}
$$

6. Hypersquares

Here we generalize some of the results of the previous section to hypersquares.

For elements $v, w \in W$ with $v \leq w$, we let $I(v,w) = \{z \in W \mid v \leq z \leq w\} \subset W$ be the interval between v and w. Analogously we define $I^{F^+}(v,w) = I(v,w)$ for $v, w \in F^+$. Note that if we have fixed reduced decompositions of $v, w \in W$, the set $I^{F^+}(v,w)$ is in general

not compatible with $I(v,w)$ in the sense that $\gamma(I^{F^+}(v,w)) = I(v,w)$. Further we set for any interval $I = I(v,w)$,

$$\mathrm{head}(I) = w \text{ and } \mathrm{tail}(I) = v.$$

Definition 6.1. Let $v \leq w \in W$ with $\ell(w) - \ell(v) = d$. We say that $I(v,w)$ is a hypersquare of dimension d in W if

$$\#\{z \in I(v,w) \mid \ell(z) = \ell(w) - i\} = \binom{d}{i}$$

for all $1 \leq i \leq d$.

If $I(v,w)$ is a hypersquare, then $\#I(v,w) = 2^d$ (the converse is also true). In this case we also write $Q(v,w) = I(v,w)$.

The definition of a hypersquare in F^+ is similar but easier in the sense that for all $v, w \in F^+$ with $v \preceq w$ the cardinality of $I(v,w)$ is always $2^{\ell(w)-\ell(v)}$.

Definition 6.2. Let $v \preceq w \in F^+$ with $\ell(w) - \ell(v) = d$. The associated hypersquare of dimension d in F^+ is given by the set $Q(v,w) = I(v,w)$.

If we consider $v, w \in W$ with $v \leq w$ and with fixed reduced decompositions $w = s_{i_1} \cdots s_{i_r}$ and $v = s_{j_1} \cdots s_{j_s}$, then we also write $Q^{F^+}(v,w)$ for $Q(s_{j_1} \cdots s_{j_s}, s_{i_1} \cdots s_{i_r})$. For a hypersquare resp. interval $I = I(v,w)$ of W (resp. F^+), let

$$X(v,w) := X(I) := \bigcup_{w \in I} X(w)$$

be the induced locally closed subvariety of X (resp. of $X^{\ell(w)+1}$ where $w = \mathrm{head}(I)$). In particular, for $w \in F^+$ the compactification $\overline{X}(w)$ of $X(w)$ can be rewritten as

$$\overline{X}(w) = X(Q(1,w)).$$

Lemma 6.3. *Let $Q \subset W$ (resp. $Q \subset F^+$) be a hypersquare. Then the variety $X(Q)$ is smooth.*

Proof. If $Q \subset F^+$, then the claim follows from Proposition 3.13 since $X(Q)$ is an open subset of $\overline{X}(\mathrm{head}(Q))$. If $Q \subset W$, then the claim follows from [BL, Theorem 6.2.10]. Indeed, let $d := \dim Q$. By the rigidity of Q it has to coincide with the Bruhat graph $B(\mathrm{tail}(Q), \mathrm{head}(Q))$. Then loc.cit. says that that $X(Q)$ is (rationally) smooth if each vertex in the Bruhat graph has exactly d edges. But each vertex in Q has already by definition d edges. □

Definition 6.4. A square $Q \subset W$ (resp. $Q \subset F^+$) is called special if it has the shape $Q = \{sw's, sw', w's, w'\}$ for some $w, w' \in W$, $s \in S$ (resp. F^+) with $\ell(w) = \ell(w') + 2$. In this case we also write $Q_w = Q_{w,s} = Q$.

The generalization of Proposition 4.5 is given by the next result.

Proposition 6.5. *Let $Q' = Q(v', w') \subset W$ be a hypersquare of dimension d. Suppose that for $s \in S$, the sets $sQ', Q's$ and $Q := sQ's$ are hypersquares, as well, and that $\ell(sw's) = \ell(w') + 2$, $\ell(sv's) = \ell(v') + 2$. Then $X(Q) \cup X(sQ')$ is an $\mathbb{A}^1$-bundle over $X(Q's) \cup X(Q')$. Consequently,*

$$H_c^i(X(Q) \cup X(sQ')) \cong H_c^{i-2}(X(Q's) \cup X(Q'))(-1).$$

Moreover $X(Q) \cup X(sQ') \cup X(Q's) \cup X(Q') = X(Q(v', sw's))$ is a $\mathbb{P}^1$-bundle over $X(Q's) \cup X(Q')$.

Proof. The claim follows easily from the fact the hypersquare $Q \cup sQ' \cup Q's \cup Q'$ is paved by special squares. □

Remark 6.6. The same statement is true if we work in F^+ where the assumption is automatically satisfied. In particular, if $w = sw's \in F^+$, then

$$H^i(\overline{X}(w)) = H^i(\overline{X}(w's)) \oplus H^{i-2}(\overline{X}(w's))(-1)$$

for all $i \geq 2$ and $H^0(\overline{X}(w)) = H^0(\overline{X}(w's))$. Analogously, we have

$$H^i(\overline{X}(w)) = H^i(\overline{X}(sw')) \oplus H^{i-2}(\overline{X}(sw'))(-1)$$

for all $i \geq 2$ and $H^0(\overline{X}(w)) = H^0(\overline{X}(sw'))$.

Remark 6.7. An analogue of the above formula is already formulated in [DMR, Prop. 3.2.3] in the case of Weyl group elements.

For later use we mention the following statement. Recall that we denote for any Γ-module V and any integer i by $V\langle i\rangle$ the eigenspace of the arithmetic Frobenius with eigenvalue q^i.

Lemma 6.8. *Let $w = sw's \in F^+$ with $\mathrm{ht}(sw') = 0$. Then $H_c^{2i+1}(X(s^2, w))\langle -i\rangle = 0$.*

Proof. If $\ell(w) > \ell(\gamma(w))$ then s commutes with every simple reflection in w'. Hence $X(s^2, w)$ is homeomorphic to $X(s^2) \times \overline{X}(w')$. One computes easily that

$$H_c^*(X(s^2)) = i_B^G / i_{P(s)}^G[-2] \bigoplus i_{P(s)}^G(-2)[-4].$$

Further the cohomology of $\overline{X}(w')$ vanishes in odd degree by Proposition 3.4, thus $H_c^{2i+1}(X(s^2, w)) = (0)$ by the Künneth formula.

So let $\ell(w) = \ell(\gamma(w))$ and suppose that $V = j_\lambda(-i) \subset H_c^{2i+1}(X(s^2,w)) \neq 0$. The complement of $X(w)$ in $X(s^2,w)$ is just the union $\bigcup_{v' \prec w'} X(s^2, sv's)$. Moreover, the intersection of two subsets of the form $X(s^2,sv's)$ has again such a shape. Consider the induced spectral sequence $E_1^{p,q} \Longrightarrow H_c^{p+q}(X(w))$ with

$$E_1^{p,q} = \bigoplus_{\substack{\{v_1',\ldots,v_p'\} \\ v_i' \prec w', \ell(v_i') = \ell(w')-1}} H_c^q\Big(\bigcap_{i=1}^p X(s^2, sv_i's)\Big)$$

for $p \geq 1$ and $E_1^{0,q} = H_c^q(X(s^2,sw's))$. By induction we deduce that V has to be induced by $H_c^{2i+1}(X(w))$. Further we may assume that w is full. By Corollary 5.10 and by Proposition 5.3 it follows that $i = \ell(w) - 2$.

1. Case: $w = s_1s_2s_3\cdots s_{n-1}s_1$ (or $w = s_{n-1}s_1s_2\cdots s_{n-2}s_{n-1}$ etc.) i.e., $s = s_1$ or $s = s_{n-1}$.

By Corollary 5.10 we conclude that $j_\lambda = j_{(n-2,2)}$. We consider the square

$$Q = \{w, sv_1's, sv_2's, su's\} \subset F^+$$

with

$$\begin{aligned} v_1' &= s_2s_3\cdots s_{n-3}s_{n-2}, \\ v_2' &= s_2s_3\cdots s_{n-3}s_{n-1}, \\ u' &= s_2s_3\cdots s_{n-3}. \end{aligned}$$

Now $H_c^{2i}(X(su's)) = H_c^{2\ell(su's)}(X(su's)) = i_{P(su's)}^G(-i) = i_{P_d}^G(-i)$ with $d = (n-2,1,1) \in \mathcal{D}$. It is enough to see that the $j_{(n-2,2)}$-isotypic part in $H_c^{2i+1}(X(Q))$ vanishes. For this we consider the boundary map $H_c^{2i}(X(sv_2's) \cup X(su's)) \longrightarrow H_c^{2i+1}(X(sv_1's) \cup X(w))$ and moreover the extended boundary map

$$H_c^{2i}(X(sv_2's) \cup X(su's) \cup X(sv_2') \cup X(su')) \longrightarrow H_c^{2i+1}(X(w) \cup X(sv_1's) \cup X(sv_1') \cup X(sw'))$$

which identifies with

$$H_c^{2i-2}(X(v_2's) \cup X(u's) \cup X(v_2') \cup X(u'))(-1) \longrightarrow H_c^{2i-1}(X(w's) \cup X(v_1's) \cup X(v_1') \cup X(w'))(-1).$$

By weight reasons we deduce that

$$H_c^{2i}(X(sv_2's) \cup X(su's)) \subset H_c^{2i}(X(sv_2's) \cup X(su's) \cup X(sv_2') \cup X(su')).$$

On the other hand, $V \not\subset H_c^{2i}(X(sv_1') \cup X(sw'))$ as $H_c^{2i}(X(sv_1')) = i_{P_{(n-1,1)}}^G(-i)$ and by Proposition 3.3. Hence it suffices to see that V does not appear in the cokernel of the extended boundary map. Now

$$V(-1) \subset H_c^{2i-2}(X(u's)) = i_{P(u's)}^G(-i+1) = i_{P_{(n-2,1,1)}}^G(-i+1)$$

resp.

$$V(-1) \subset H_c^{2i-2}(X(v_2')) = i^G_{P(v_2')}(-i+1)) = i^G_{P_{(1,n-3,2)}}(-i+1).$$

On the other hand,

$$V(-1) \subset H_c^{2i-1}(X(w')) = i^G_{P_{(1,n-2,1)}}/i^G_{P_{(1,n-1)}}(-i+1),$$

$$V(-1) \subset H_c^{2i-1}(X(v_1's)) = i^G_{P_{(n-2,1,1)}}/i^G_{P_{(n-1,1)}}(-i+1)$$

and

$$V(-1) \subset H_c^{2i-2}(X(v_1')) = i^G_{P_{(1,n-2,1)}}(-i+1).$$

The result follows now easily by intertwining arguments as the contribution $V(-1) \subset H_c^{2i-2}(X(v_1'))$ maps diagonally to $H_c^{2i-1}(X(v_1's)) \bigoplus H_c^{2i-1}(X(w'))$ and $H_c^{2i-2}(X(v_2'))$ maps surjectively onto $H_c^{2i-1}(X(w'))$.

2. Case: $w = s_i s_1 s_2 \cdots \widehat{s_i} \cdots s_{n-1} s_i$ with $2 \leq i \leq n-2$.

By Corollary 3.8 we see that $H_c^{2i+1}(X(w))$ is a direct sum of quotients of induced representations $i^G_P(-i)$ where P is not a proper maximal subgroup. But $H_c^{2i}(X(sw'))$ does not kill $H_c^{2i+1}(X(w))$ by Prop. 5.9. Further $H_c^{2i}(X(sv')) = H_c^{2\ell(w)-4}(X(sv')) = H_c^{2\ell(sv')}(X(sv')) = i^G_{P(sv')}(-i)$ for all $v' \prec w'$ with $\ell(v') = \ell(w') - 1$, where $P(sv') \subset G$ is a proper maximal subgroup. Hence the representations of the first kind have to be killed by weight reasons by some $H_c^{2i}(X(su's))$ with $\ell(u') = \ell(w') - 2$. The claim follows. $\square$

7. Some further results on the cohomology of DL-varieties

We shall proof some of the statements given in the introduction. We start with the analogue version of Theorem 4.1 for elements in B^+.

Lemma 7.1. *Let $w \in B^+$ such that $\ell(\beta(w)) < \ell(w)$, i.e. such that $w \in B^+ \setminus W$. Then w has the shape $w = w_1 \cdot s \cdot s \cdot w_2$ for some $s \in S$ and $w_1, w_2 \in B^+$.*

Proof. See [GKP, Exercise 4.1]. $\square$

We extend the definition of the height function on W to F^+.

Definition 7.2. i) Let $w \in B^+$. We define the height inductively by

$$\mathrm{ht}(w) := \begin{cases} \mathrm{ht}(w) & \text{if } w \in W \\ \\ h(w_1 w_2) + 1 & \text{if } w = w_1 \cdot s \cdot s \cdot w_2 \text{ is as above} \end{cases}$$

ii) For $w \in F^+$, we set $\mathrm{ht}(w) := \mathrm{ht}(\alpha(w))$.

Thus we may write all elements in B^+ modulo cyclic shift in the shape $w = sw's$ for some $s \in S$ and $w' \in B^+$. In particular, for $w \in F^+$, there is always an element $sw's \in F^+$ with $\ell(w) = \ell(sw's)$ and with $H_c^*(X(w)) = H_c^*(X(sw's))$. We shall use this property to prove the next statement.

Lemma 7.3. *Let $w \in F^+$. Then $H^i(\overline{X}(w)) = 0$ for i odd.*

Proof. Since $\overline{X}(w)$ is smooth and projective it suffices to show that all eigenvalues of the Frobenius on the cohomology groups $H^*(\overline{X}(w))$ are integral powers of q. By considering the spectral sequence to the stratification $\overline{X}(w) = \bigcup_{v \preceq w} X(v)$ it suffices to show that this property is valid for the cohomology groups $H_c^*(X(v))$. By what we have said above, we may suppose that $v = sv's$ for some $s \in S$ and $v' \in F^+$. By induction on the length we know that the assertion is true for $H_c^*(X(v's))$ and $H_c^*(X(v'))$, hence for $H_c^*(X(v's) \cup X(v'))$. But $X(v) \cup X(sv')$ is an $\mathbb{A}^1$-bundle over $X(v's) \cup X(v')$ by Remark 4.14. Thus the assertion is true for $H_c^*(X(v) \cup X(sv'))$. Finally, by considering again the corresponding long exact cohomology sequence the claim follows. $\square$

Let $X(w)$ be a DL-variety attached to an element $w \in W$ and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of w. In order to compute the cohomology of $X(w) \cong X^{F^+}(w)$ we consider the stratification $\overline{X}(w) = \bigcup_{v \preceq w} X^{F^+}(v)$ in which $X(w)$ appears as an open stratum. Write

$$\overline{X}(w) = X^{F^+}(w) \,\dot{\cup}\, Y$$

where $Y = \bigcup_{\substack{v \prec w \\ \ell(v) = \ell(w)-1}} \overline{X}(v)$. We consider the induced spectral sequence

$$E_1^{p,q} \Longrightarrow H_c^{p+q}(X^{F^+}(w))$$

with

$$E_1^{p,q} = \bigoplus_{\substack{\{v_1, \ldots, v_p\} \\ v_i \prec w, \ell(v_i) = \ell(w)-1}} H_c^q\Big(\bigcap_{i=1}^p \overline{X}(v_i)\Big)$$

for $p \geq 1$ and $E_1^{0,q} = H^q(\overline{X}(w))$. Note that the intersection $\bigcap_1^p \overline{X}(v_i)$ is nothing else but $\overline{X}(v)$ where $v \in F^+$ is the unique element of length $\ell(w) - p$ with $v \preceq v_i$, $i = 1, \ldots, p$.

Remark 7.4. The element v could be considered as the greatest common divisor or the meet of the elements $v_1, \ldots, v_p$. In fact, the set $Q(1, w)$ is a bounded distributive lattice.

Hence the ith row of E_1 is given by the complex

$$0 \longrightarrow H^i(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{v\prec w \\ \ell(v)=\ell(w)-1}} H^i(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{v\prec w \\ \ell(v)=\ell(w)-2}} H^i(\overline{X}(v)) \longrightarrow \cdots \tag{7.1}$$

We shall analyse this spectral sequence. As all varieties $\overline{X}(v)$ are smooth and projective their cohomology is pure. We conclude that $E_2 = E_\infty$ and hence by weight reasons that

$$H^i_c(X(w)) = \bigoplus_{p+q=i} E_2^{p,q}.$$

Proposition 7.5. *The representations $H^*(\overline{X}(w))$ and $H^*_c(X(w))$ are Frobenius semisimple for all $w \in F^+$.*

Proof. Again the proof is by induction on $\ell(w)$. The start of induction is given by Proposition 3.4. As the weights of $H^{i-1}(Y)$ are different from $H^i(\overline{X}(w))$, it is enough to prove that both of these objects are Frobenius semisimple. But by considering the E_2-term of the obvious spectral sequence converging to the cohomology of Y and by induction hypothesis it suffices to show that $H^i(\overline{X}(w))$ is Frobenius semisimple.

Since $\overline{X}(w)$ is smooth and projective we get by Poincaré duality the identity $H^i(\overline{X}(w)) = H^{2\ell(w)-i}(\overline{X}(w))(-\ell(w)+i)$ for $i \leq \ell(w)$. So it suffices to consider the case $i \leq \ell(w)$. Further we know that $H^i_c(X(w)) = (0)$ for all $i < \ell(w)$, cf. Remark 5.11. Hence we deduce that $H^i(\overline{X}(w)) \subset H^i(Y)$ for all $i < \ell(w)$. In the latter case the claim follows by induction considering again the spectral sequence to Y.

If $i = \ell(w)$ (is even), then we consider the long exact sequence

$$\begin{aligned} 0 \longrightarrow \quad H^{\ell(w)-1}(Y) \quad &\longrightarrow H^{\ell(w)}_c(X(w)) \longrightarrow H^{\ell(w)}(\overline{X}(w)) \\ \longrightarrow \quad H^{\ell(w)}(Y) \quad &\longrightarrow H^{\ell(w)+1}_c(X(w)) \longrightarrow 0. \end{aligned}$$

We claim that if there is some irreducible subrepresentation $V = j_\lambda(-j) \subset H^{\ell(w)}_c(X(w))$, then $j < \frac{i}{2}$. Here we may suppose that $w = sw's$. If $V \subset H^{\ell(w)-1}(Y)$, then the claim follows by weight reasons. If $V \subset H^{\ell(w)}(\overline{X}(w))$, i.e. $j = \frac{i}{2}$, then it is in the kernel of the map

$$H^{\ell(w)}(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{v\prec w \\ \ell(v)=\ell(w)-1}} H^{\ell(w)}(\overline{X}(v)).$$

Since $w's$ appears as index in this direct sum, the kernel is by Remark 6.6 the same as the kernel of the map

$$H^{\ell(w)-2}(\overline{X}(w's))(-1) \longrightarrow H^{\ell(w)}(\overline{X}(sw')) \bigoplus \bigoplus_{\substack{v'\prec w's \\ \ell(v')=\ell(w')-1}} H^{\ell(w)-2}(\overline{X}(v's))(-1).$$

In particular, it is contained in the kernel of the map

$$H^{\ell(w)-2}(\overline{X}(w's))(-1) \longrightarrow \bigoplus_{\substack{v' \prec w's \\ \ell(v')=\ell(w')-1}} H^{\ell(w)-2}(\overline{X}(v's))(-1).$$

Since the contribution of w' is missing on the RHS, we deduce that

$$V(1) = j_\lambda(-j+1) \subset H_c^{\ell(w)-2}(X(w's) \cup X(w')).$$

But $H_c^{\ell(w)-2}(X(w's)) = (0)$, as $\ell(w') = \ell(w) - 2 < \ell(w's)$. Hence $V(1) \subset H_c^{\ell(w')}(X(w'))$. Again by induction we know that $j - 1 < \frac{\ell(w')}{2}$. But $\frac{\ell(w')}{2} = \frac{\ell(w)-2}{2} = \frac{\ell(w)}{2} - 1$. Hence we get a contradiction. □

Corollary 7.6. *(of the proof) Let $w \in F^+ \setminus \{e\}$ and let $V = j_\lambda(-i) \subset H_c^{\ell(w)}(X(w))$ for some $\lambda \in \mathcal{P}$. Then $2i < \ell(w)$.*

Remark 7.7. The latter result is proved in [DMR, Prop. 3.3.31 (iv)] for arbitrary reductive groups.

We further may deduce the following vanishing result.

Proposition 7.8. *Let $w \in F^+$ with $\mathrm{ht}(w) \geq 1$. Then $H_c^{2\ell(w)-1}(X(w)) = 0$.*

Proof. By Corollary 5.10 we may suppose that $\mathrm{ht}(sw') \geq 1$. We consider the long exact cohomology sequence

$$\begin{aligned} \cdots \longrightarrow \quad & H_c^{2\ell(w)-2}(X(w) \cup X(sw')) \quad \xrightarrow{r} H_c^{2\ell(w)-2}(X(sw')) \longrightarrow H_c^{2\ell(w)-1}(X(w)) \\ \longrightarrow \quad & H_c^{2\ell(w)-1}(X(w) \cup X(sw')) \quad \longrightarrow \cdots . \end{aligned}$$

The map $r = r_{w,sw'}^{2\ell(w)-2}$ has to be surjective since $H_c^{2\ell(w)-2}(X(sw')) = H_c^{2\ell(sw')}(X(sw')) = i_G^G(-\ell(sw'))$ is the top cohomology group. On the other hand, by Proposition 4.5 we know that $H_c^{2\ell(w)-1}(X(w) \cup X(sw')) = H_c^{2\ell(w)-3}(X(w's) \cup X(w'))(-1)$. But

$$H_c^{2\ell(w)-3}(X(w's)) = H_c^{2\ell(w's)-1}(X(sw')) = 0$$

by induction. Further $H_c^{2\ell(w)-3}(X(w')) = H_c^{2\ell(w')+1}(X(w')) = 0$. Hence we conclude the claim. □

This vanishing result has the following consequences.

Corollary 7.9. *Let $w \in F^+$ with $\mathrm{ht}(w) \geq 1$. Then*

$$\begin{aligned} H^{2\ell(w)-2}(\overline{X}(w)) &= H_c^{2\ell(w)-2}(X(w)) \bigoplus \bigoplus_{\substack{v \preceq w \\ \ell(v)=\ell(w)-1}} H_c^{2\ell(w)-2}(X(v)) \\ &= H_c^{2\ell(w)-2}(X(w)) \bigoplus \bigoplus_{\substack{v \preceq w \\ \ell(v)=\ell(w)-1}} i_{P(v)}^G(-(\ell(w)-1)). \quad \square \end{aligned}$$

Corollary 7.10. *Let $w = sw's \in F^+$ with $\mathrm{ht}(w') \geq 1$ and $\mathrm{supp}(w) = S$. Then*

$$H_c^{2\ell(w)-2}(X(w)) = H_c^{2\ell(w's)-2}(X(w's))(-1) \bigoplus (i^G_{P(w')} - i^G_G)(-\ell(w)+1).$$

Proof. Since $\mathrm{ht}(sw') \geq 1$ we have an exact sequence

$$0 \longrightarrow H_c^{2\ell(w)-2}(X(w)) \longrightarrow H_c^{2\ell(w)-2}(X(w) \cup X(sw')) \longrightarrow H_c^{2\ell(w)-2}(X(sw')) \longrightarrow 0.$$

But the assumption $\mathrm{supp}(w) = S$ also implies that $\mathrm{supp}(sw') = S$. Hence we get

$$H_c^{2\ell(w)-2}(X(sw')) = H_c^{2\ell(sw')}(X(sw')) = i^G_G(-\ell(w)+1).$$

Further we have by Corollary 4.5 $H_c^{2\ell(w)-2}(X(w) \cup X(sw')) = H_c^{2\ell(w)-4}(X(w's) \cup X(w'))$. But since $\mathrm{ht}(w') \geq 1$ we deduce that

$$H_c^{2\ell(w)-5}(X(w')) = H_c^{2\ell(w')-1}(X(w')) = 0.$$

Moreover, $H_c^{2\ell(w)-3}(X(w's)) = H_c^{2\ell(w's)-1}(X(w's)) = 0$. Now the result follows easily. $\square$

We reconsider the spectral sequence

$$E_1^{p,q} \Rightarrow H^{p+q}(H^*(\overline{X}(w))).$$

which is induced by the stratification $\overline{X}(w) = \dot{\bigcup}_{v \preceq w} X(v)$, The qth line in the E_1-term is the complex

$$\bigoplus_{\substack{v' \preceq w \\ \ell(v')=q}} H_c^{2q}(X(v')) \longrightarrow \cdots \longrightarrow \bigoplus_{\substack{v' \preceq w \\ \ell(v')=q+j}} H_c^{2q+j}(X(v')) \longrightarrow \bigoplus_{\substack{v \preceq w \\ \ell(v)=q+j+1}} H_c^{2q+j+1}(X(v)) \longrightarrow \cdots$$

where the homomorphisms are induced by the boundary maps $\delta_{v',v}$.

`Picture`:

$$\begin{array}{cccccccc}
 & & & & & & \vdots & \vdots \\
 & & H^4 & H^5 & H^6 & \cdots & H^{i+2} & H^{i+3}\cdots \\
 & H^2 & H^3 & H^4 & H^5 & \cdots & H^{i+1} & H^{i+2}\cdots \\
H^0 & H^1 & H^2 & H^3 & H^4 & \cdots & H^i & H^{i+1}\cdots \\
\hline
e & \ell=1 & \ell=2 & \ell=3 & \ell=4 & \cdots & \ell=i & \ell=i+1
\end{array}$$

Of course this spectral sequence degenerates and we may write by weight reasons and by Proposition 7.5 for all $0 \leq i \leq \ell(w)$,

$$H^{2i}(\overline{X}(w)) = \bigoplus_{j=i}^{\ell(w)} H_c^{2i}(X(w)(j))'$$

where

$$X(w)(j) := \bigcup_{\substack{v \preceq w \\ \ell(v)=j}} X(v)$$

and where $H_c^{2i}(X(w)(j))' \subset H_c^{2i}(X(w)(j)) = \bigoplus_{\substack{v \preceq w \\ \ell(v)=j}} H_c^{2i}(X(v))$.

For $v \preceq w$ with $\ell(v) = i$, we have $H_c^{2i}(X(v)) = H^{2i}(\overline{X}(v)) = i_{P(v)}^G(-i)$. Here $P(v) \subset G$ is the std psgp attached to v, cf. (2.5.) By Remark 4.9 the trivial representation does appear in the top cohomology degree of a DL-variety. Hence the subrepresentation $i_G^G(-i) \subset i_{P(v)}^G(-i)$ survives the spectral sequence. Thus there is grading

$$H^{2i}(\overline{X}(w)) = \bigoplus_{\substack{z \preceq w \\ \ell(v)=i}} H(w)_z$$

with $H(w)_z \supset i_G^G(-i)$ for certain representations $H(w)_z$.

In the sequel we shall see that we may suppose that the objects $H(w)_z$ are induced representations. Recall that the following three operations C, K, R on elements in F^+ allows us to transform an arbitrary element $w \in F^+$ into the shape $w = sw's$ with $s \in S$ and $w' \in F^+$.

(I) (Cyclic shift) If $w = sw'$ with $s \in S$, then we set $C(w) = w's$.

(II) (Commuting relation). If $w = w_1 stw_2$ with $s, t \in S$ and $st = ts$. Then we set $K(w) = w_1 tsw_2$.

(III) (Replace sts by tst) If $w = w_1 stsw_2$ with $s, t \in S$ and $sts = tst$. Then we set $R(w) = w_1 tstw_2$.

We shall analyse the induced behaviour on the cohomology of Demazure varieties. In [DMR] the following generalization of Proposition 4.2 is proved.

Proposition 7.11. *Let $w = s_{i_1} \cdots s_{i_r} \in F^+$. Then for all $i \geq 0$, there is an isomorphism of H-modules*

$$H_c^i(X(w)) \longrightarrow H_c^i(X(C(w))).$$

Proof. This is a special case of [DMR, Proposition 3.1.6]. □

As a cyclic shift does not affect the cohomology. The natural question which arises is if the same holds true for the compactification $\overline{X}(w)$. The following statement gives a positive answer.

Proposition 7.12. *Let $w = s_{i_1} \cdots s_{i_r} \in F^+$. Then for all $i \geq 0$, there is an isomorphism of H-modules*

$$H^i(\overline{X}(w)) \longrightarrow H^i(\overline{X}(C(w))).$$

Proof. Set $z := C(w)$. By definition of $\overline{X}(w)$ we have a covering $\overline{X}(w) = \bigcup_{v \preceq w} X(v)$ and similar for $\overline{X}(z)$. By considering the spectral sequence to these coverings and by the fact that the cyclic shift class does not affect the cohomology of a stratum $X(v)$ it is enough to see that the sets $Q(1,z)$ and $\{C(v) \mid v \in Q(1,w)\}$ are the same. So let $v \preceq w$. If $v \preceq s_{i_1}s_{i_2}\cdots s_{i_{r-1}}$, then obviously $v \preceq z$. If $v = v' \cdot s_{i_r}$, then $C(v) = s_{i_r}v' \in Q(1,z)$. The claim follows. $\square$

Corollary 7.13. *let $sw's \in F^+$. Then $H^i(\overline{X}(sw')) = H^i(\overline{X}(w's))$ for all $i \geq 0$.* $\square$

Remarks 7.14. i) Lemma 7.12 is already proved in a more general situation in [DMR, Prop. 3.1.6].

ii) Consider the equivalence relation $\sim$ on F^+ leading to cyclic shift classes in the sense of [GP], i.e., $v, w \in F^+$ are equivalent if there is some integer $i \geq 0$ with $C^i(w) = v$. Thus we can associate to any element in $C^+ = F^+/\sim$ its cohomology. Also the height function ht on F^+ depends only on the cycle shift class. Sometimes it is useful to interpret in what follows the image of an element $w = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}s_{i_r}$ in C^+ as a circle, i.e.,

$$\begin{array}{ccc}
 & s_{i_1} & \\
s_{i_r} & & s_2 \\
\vdots & & \vdots \\
s_{i_{j+1}} & & s_{i_{j-1}} \\
 & s_{i_j} &
\end{array}$$

As we have observed in Lemma 7.12 the Cyclic shift operator does not affect the cohomology of a Demazure variety. The same holds true for operation (II).

Proposition 7.15. *Let $w = w_1stw_2 \in F^+$ with $s, t \in S$ and $st = ts$. Set $K(w) = w_1tsw_2$. Then $H^i(\overline{X}(w)) = H^i(\overline{X}(K(w)))$ for all $i \geq 0$.*

Proof. The reason is that the stratifications of the varieties $\overline{X}(w)$ and $\overline{X}(K(w))$ are essentially the same in the obvious sense. $\square$

Let $w = w_1stsw_2 \in F^+$. In the sequel we also write $w = w_1s_lts_rw_2$ (l for left, r for right) in order to distinguish the reflection s in its appearance in w. Further we write $w_1s^2w_2$ for the subword $w_1s_ls_rw_2$.

Proposition 7.16. *Let $s, t \in S$, $w_1, w_2 \in F^+$ and set $w = w_1stsw_2$ and $v = R(w) = w_1tstw_2$. Then for all $i \geq 0$,*

$$H^i(\overline{X}(v)) = H^i(\overline{X}(w)) - H^i_c(X(s_l, w_1s^2w_2)) + H^i_c(X(t_r, w_1t^2w_2))$$

$$= H^i(\overline{X}(w)) - H^{i-2}(\overline{X}(w_1sw_2))(-1) + H^{i-2}(\overline{X}(w_1tw_2))(-1).$$

(Here we mean as before when writing $-H^{i-2}(\overline{X}(w_1sw_2))(-1)$ that $H^{i-2}(\overline{X}(w_1sw_2))(-1)$ appears as a submodule in $H^i(\overline{X}(w))$.)

Proof. The varieties $X(w)$ and $X(v)$ differ by the locally closed subsets $X(s, w_1s^2w_2) \subset \overline{X}(w)$ resp. $X(t, w_1t^2w_2) \subset \overline{X}(v)$, i.e., the constructible subsets $\overline{X}(w) \setminus X(s, w_1s^2w_2)$ and $\overline{X}(v) \setminus X(t, w_1t^2w_2)$ have homeomorphic stratifications. Hence we see that the Euler-Poincaré characteristics in the Grothendieck group of G-modules are the same. More precisely, we have

$$\mathrm{EP}(\overline{X}(w)) - \mathrm{EP}(X(s, w_1s^2w_2)) = \mathrm{EP}(\overline{X}(v)) - \mathrm{EP}(X(t, w_1t^2w_2)),$$

where we set for a variety X, $\mathrm{EP}(X) = \sum_i (-1)^i H^i_c(X)$. Moreover, the variety $X(s_l, w_1s^2w_2)$ (resp. $X(t_r, w_1t^2w_2)$) is a $\mathbb{A}^1$-bundle over $\overline{X}(w_1sw_2)$ (resp. $\overline{X}(w_1tw_2)$). Hence the individual cohomology groups of the varieties vanish in odd degree. It follows immediately that for all $j \geq 0$, the above identity $H^j(\overline{X}(w)) - H^j_c(X(s_l, w_1s^2w_2)) = H^j(\overline{X}(v)) - H^j_c(X(t_r, w_1t^2w_2))$ holds true in the Grothendieck group of H-representations.

It suffices therefore to show that $H^{i-2}(\overline{X}(w_1sw_2))(-1) \subset H^i(\overline{X}(w))$. Here we use induction on the length of elements in F^+ together with the number of necessary operations (I) - (III) to transform w into an element of the shape $rw'r$ for some $r \in S$. The operations (I) and (II) are easy to handle by Propositions 7.12, 7.15. Suppose now that $w_2 = w_3rqrw_4$ with $r, q \in S, rq \neq qr$, and that the inclusion holds for $y = w_1stsw_3qrqw_4$, i.e., $H^{i-2}(\overline{X}(w_1sw_3qrqw_4))(-1) \subset H^i(\overline{X}(y))$. By induction on the length we deduce that $H^{i-4}(\overline{X}(w_1sw_3qw_4))(-1) \subset H^{i-2}(\overline{X}(w_1sw_3qrqw_4))$ and we may write

$$\begin{aligned} H^{i-2}(\overline{X}(w_1sw_2)) &= H^{i-2}(\overline{X}(w_1sw_3qrqw_4)) - H^{i-4}(\overline{X}(w_1sw_3qw_4))(-1) \\ &\quad + H^{i-4}(\overline{X}(w_1sw_3rw_4))(-1). \end{aligned}$$

Further the identity

$$H^i(\overline{X}(w)) = H^i(\overline{X}(y)) - H^{i-2}(\overline{X}(w_1stsw_3qw_4))(-1) + H^{i-2}(\overline{X}(w_1stsw_3rw_4))(-1)$$

is satisfied by induction on the number of necessary operations (and with respect to the element $w_1stsw_3qw_4$). Again by induction on the length we have $H^{i-4}(\overline{X}(w_1sw_3rw_4))(-1) \subset H^{i-2}(\overline{X}(w_1stsw_3rw_4))$ and $H^{i-4}(\overline{X}(w_1sw_3qw_4))(-1) \subset H^{i-2}(\overline{X}(w_1stsw_3qw_4))$. Hence we see that $H^{i-2}(\overline{X}(w_1sw_2))(-1) \subset H^i(\overline{X}(w))$.

If on the other hand the subword sts is involved in the above replacement then the formula in the claim performs automatically the induction step. So we may suppose that $w = rw'r$ with $r \in S$.

1) Case. If $sts \preceq w'$ then the claim follows by induction on $\ell(w)$. Indeed, we have $H^{2i}(\overline{X}(w)) = H^{2i}(\overline{X}(w'r)) \oplus H^{2i-2}(\overline{X}(w'r))(-1)$. Now apply the induction hypothesis to the element $w'r \in F^+$, etc.

2) Case. Let $r = s$ and $w = sw''sts$ for some subword $w'' \prec w'$. Here we proceed as in Case 1.

3) Case. $r = s$ and $w = stw''s$ for some subword $w'' \prec w'$ such that s commutes with every element in w''. As we may assume that w has full support, it follows that $s = (1, 2)$ and $t = (2, 3)$ (or that $s = (n-1, n)$ and $t = (n-2, n-1)$) Moreover we may suppose that no simple reflection appears twice in w' since then we could produce by the operations (I) - (III) an element uzu with $u \in S$ such that sts appears in z. Hence we could apply Case 1 again.

So let w' be multiple free. Set $Y' := \overline{X}(sw's) \subset \overline{X}(w)$ and $Y := \overline{X(\gamma(w's))} \subset X$, $U' := \overline{X}(w) \setminus \overline{X}(sw's)$, $U := \overline{X(\gamma(w))} \setminus Y$. The natural proper map $\overline{X}(w) \longrightarrow \overline{X(\gamma(w))}$ induces a commutative diagram of long exact cohomology sequences

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H^i_c(U') & \longrightarrow & H^i(\overline{X}(w)) & \longrightarrow & H^i(Y') & \longrightarrow & H^{i+1}_c(U') & \longrightarrow \cdots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \cdots \longrightarrow & H^i_c(U) & \longrightarrow & H^i(\overline{X(\gamma(w))}) & \longrightarrow & H^i(Y) & \longrightarrow & H^{i+1}_c(U) & \longrightarrow \cdots \end{array} \tag{7.2}$$

By the special shape of w, we see that $\overline{X(\gamma(w))}$ is smooth. Hence we deduce by weight reasons that $H^{i+1}(\overline{X(\gamma(w))}) = 0$. Thus the boundary map $H^i(Y) \longrightarrow H^{i+1}_c(U)$ is surjective. As $H^i(Y') = H^i(Y) \oplus H^{i-2}(Y)(-1)$ the claim follows. □

Theorem 7.17. *Let $w \in F^+$. The cohomology of $\overline{X}(w)$ in degree $2i$ can be written as*

$$H^{2i}(\overline{X}(w)) = \bigoplus_{\substack{z \preceq w \\ \ell(v)=i}} H(w)_z$$

with $H(w)_z = i^G_{P^w_z}(-i)$ for certain standard parabolic subgroups $P^w_z \subset G$.

Proof. The proof is by double induction on n and $\ell(w)$. The start of the exterior induction is given by $n = 1, 2$. Here the situation is trivial. By Remark 3.15 we may suppose that the start of the the interior induction is given by an element with full support. So let w be a minimal element with full support, i.e. w is a Coxeter element. Then the assertion (i) follows from Proposition 3.4. Now let $\ell(w) \geq h + 1$. Then $h(w) \geq 1$. If $w = sw's$, then the assertion follows from Remark 3.15. More precisely, by induction we may write

$$H^*(\overline{X}(w's)) = \bigoplus_{v \preceq w's} i^G_{P_v^{w's}}(-\ell(v))[-2\ell(v)].$$

Then

$$H^*(\overline{X}(w)) = \bigoplus_{v \preceq w} i^G_{P_v^w}(-\ell(v))[-2\ell(v)]$$

where $P_v^w = P_v^{w's}$ if $v \preceq w's$ and $P_v^w = P_{v'}^{sw'}$ if $v \in Q(sw', w)$. Here $v = sv'$.

In the general situation we must apply the operations (I) - (III) to w.

(I) Let $w = sw'$, with $s \in S$ and $w' \in F^+$. Define for $v \preceq w$,

$$C(v) = \begin{cases} v's & \text{if } v = sv' \\ v & \text{if } v \preceq w'. \end{cases}$$

The cohomology of the varieties $\overline{X}(w)$ and $\overline{X}(C(w))$ are isomorphic by Lemma 7.12. Then the assignment $H(C(w))_{C(z)} := H(w)_z$ for $\ell(z) = i$, defines the desired grading on $H^{2i}(\overline{X}(C(w)))$.

(II) Let $w = w_1stw_2$. Define for $v \preceq w$,

$$K(v) = \begin{cases} v_1tsv_2 & \text{if } v = v_1stv_2 \\ v & \text{if } st \not| v. \end{cases}$$

By Lemma 7.15 we have $H^{2i}(\overline{X}(w)) = H^{2i}(\overline{X}(K(w)))$. Then the assignment $H(K(w))_{K(z)} := H(w)_z$ for $\ell(z) = i$, defines the desired grading on $H^{2i}(\overline{X}(K(w)))$.

(III) Let $w = w_1stsw_2$ and $v = R(w) = w_1tstw_2$. Then the result follows from Proposition 7.16. More precisely, we set for $z_1 \preceq w_1$ and $z_2 \preceq w_2$,

$$R(z) = \begin{cases} z_1tstz_2 & \text{if } z = z_1stsz_2 \\ z_1t_lsz_2 & \text{if } z = z_1ts_rz_2 \\ z_1st_rz_2 & \text{if } z = z_1s_ltz_2 \\ z_1z_2 & \text{if } z = z_1z_2. \end{cases}$$

Then the assignment $H(R(w))_{R(z)} := H(w)_z$ for $z \notin Q(s_l, z_1s^2z_2)$ and $H(v)_z = H(v/t_r)_{z/t_r}$ for $z \in Q(t_r, w_1t^2w_2)$ with $\ell(z) = i$ defines the desired grading on $H^{2i}(\overline{X}(v))$. □

Remarks 7.18. i) The grading

$$H^{2i}(\overline{X}(w)) = \bigoplus_{\substack{z \preceq w \\ \ell(v)=i}} H(w)_z$$

produced above is licentious since it depends on the chosen gradings with respect to the (relative) Coxeter elements in Levi subgroups.

ii) We can use Theorem 7.17 in order to reprove the statement in Remark 5.11 concerning the appearance of the Steinberg representation in the cohomology of a DL-variety $X(w)$. Indeed, the induced representation i_B^G occurs in the spectral sequence only in the contribution $H^0(\overline{X}(e))$. Hence the G-representation v_B^G occurs by Prop. 2.10 exactly in degree $\ell(w)$.

iii) We have $H^{2i}(\overline{X}(w)) = A^i(\overline{X}(w))_{\mathbb{Q}_\ell}$ for all $i \geq 0$, where $A^i(\overline{X}(w))$ denotes the Chow group of $\overline{X}(w)$ in degree i. Moreover, for $m \geq 0$, the Tate twist $-i$ contribution of $H_c^{2i+m}(X(w))$ is Bloch's higher Chow group $CH^i(X(w), m)_{\overline{\mathbb{Q}}_\ell}$, cf. [Bl]. Indeed, in view of the spectral sequence (7.1) it suffices to see the first statement. If w is a Coxeter element, then the claim follows from Remark 3.5. In general, the methods for determining $H^{2i}(\overline{X}(w))$ work for Chow groups in the same way.

iv) In [DMR, Cor. 3.3.8] the authors determine the character of $H^{2i}(\overline{X}(w))$ as H-representation. But they do not show that the underlying representation is a direct sum of induced representations.

8. The spectral sequence revisited

In this section we reconsider the spectral sequence

$$E_1^{p,q} = \bigoplus_{v \preceq w, \ell(v)=\ell(w)-p} H^q(\overline{X}(v)) \Longrightarrow H_c^{p+q}(X(w))$$

of the previous paragraph. We shall prove the remaining theorem of the introduction. For this we need some preparational work.

Let $w = sw's \in F^+$ as before. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & H_c^{2i}(X(s^2,w)) & \longrightarrow & H_c^{2i}(X(s,w)) & \longrightarrow & H_c^{2i}(X(s,w's)) & & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & H_c^{2i}(X(s,w)) & \longrightarrow & H^{2i}(\overline{X}(w)) & \longrightarrow & H^{2i}(\overline{X}(w's)) & & \longrightarrow \cdots \\
 & & \downarrow g & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & & H_c^{2i}(X(s,sw')) & \longrightarrow & H^{2i}(\overline{X}(sw')) & \xrightarrow{f} & H^{2i}(\overline{X}(w')) & & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\tag{8.1}
$$

where the maps in this diagram are the natural ones.

As for the following result we have restrict to the integer $i = 1$ as Conjecture 3.11 is proved only in this case.

Lemma 8.1. *Let $i = 1$.*

a) We have $\ker(f) = \mathrm{Im}(g)$.

b) There are gradings $H^{2i}(\overline{X}(w's)) = \bigoplus_{\substack{z \preceq w's \\ \ell(z)=i}} H(w's)_z$ and $H^{2i}(\overline{X}(w')) = \bigoplus_{\substack{z \preceq w' \\ \ell(z)=i}} H(w')_z$ such that the map $H^{2i}(\overline{X}(w's)) \longrightarrow H^{2i}(\overline{X}(w'))$ is quasi-isomorphic to the graded one. Further the induced homomorphism $H(w's)_z \longrightarrow H(w')_z$ are injective for all $z \preceq w'$.

c) There is a grading $H_c^{2i}(X(s,w)) = H^{2i-2}(\overline{X}(w's))(-1) = \bigoplus_{\substack{z' \preceq w' \\ \ell(z')=i-1}} H(w's)_{z'}$ such that $H(w's)_{z'}(-1) = H(w)_{sz'}$.

Proof. We consider the following slightly more general situation where $v, u \in F^+$ with $u \prec v$ and $\ell(u) = \ell(v) - 1$. Write $v = v_1 r v_2$ with $u = v_1 v_2$. Hence s is replaced by r, $w's$ by v, w' by u and w by $v_1 r^2 v_2$. The proof of all statements is by induction on $\ell(v)$.

If v is a Coxeter element then the statement follows from Lemma 3.11 and Lemma 6.8. So let $\mathrm{ht}(v) \geq 1$.

1. Case. $v = sv's$ for some $s \in S$.

Hence $H^{2i}(\overline{X}(v)) = H^{2i}(\overline{X}(v's)) \bigoplus H^{2i-2}(\overline{X}(v's))(-1)$.

Subcase a). $u = v's$ (The case $u = sv'$ is symmetric to this one).

Here all the statements follow easily as $\ker(f) = H^{2i-2}(\overline{X}(v's))(-1)$ and $H^{2i}(X(r, v_1 r^2 v_2) = H^{2i-2}(\overline{X}(v)) = H^{2i-2}(\overline{X}(v's)) \bigoplus H^{2i-4}(\overline{X}(v's))(-1)$.

Subcase b). $u = su's$.

Then $H^{2i}(\overline{X}(u)) = H^{2i}(\overline{X}(u's)) \bigoplus H^{2i-2}(\overline{X}(u's))(-1)$. The statements follow now by induction with respect to the homomorphism $H^j(\overline{X}(v's)) \longrightarrow H^j(\overline{X}(u's))$ with $j \in \{2i-2, 2i\}$.

2. Case. v is arbitrary.

Then we apply the operations (I) - (III) to arrange v as in the shape as in the first case. Here we use an inner induction on the necessary operations. The operations (I) and (II) are easy to handle by Propositions 7.11 and 7.15.

So let $w = w_1stsw_2$ and $v = w_1tstw_2$ and suppose that the statements are true for w.

Subcase a). $u = w_1tsw_2$. (The case $u = w_1stw_2$ is symmetric)

Let $f_w : H^{2i}(\overline{X}(w)) \longrightarrow H^{2i}(\overline{X}(u))$ resp. $f_v : H^{2i}(\overline{X}(v)) \longrightarrow H^{2i}(\overline{X}(u))$ etc. be the restriction homomorphisms. As $X(s_l, w_1s^2w_2)$ is an open subset of $X(s_l, w)$ we see that $H^{2i}_c(X(s_l, w_1s^2w_2)) \subset \ker(f_w)$. Similarly, $H^{2i}_c(X(t_r, w_1t^2w_2)) \subset \ker(f_v)$. Hence we may write

$$\ker(f_w) = H^{2i}_c(X(s_l, w_1s^2w_2)) \bigoplus \ker\big(H^{2i}(\overline{X}(w)) - H^{2i}_c(X(s_l, w_1s^2w_2)) \longrightarrow H^{2i}(\overline{X}(u))\big)$$

and

$$\ker(f_v) = H^{2i}_c(X(t_r, w_1t^2w_2)) \bigoplus \ker\big(H^{2i}(\overline{X}(v)) - H^{2i}_c(X(t_r, w_1t^2w_2)) \longrightarrow H^{2i}(\overline{X}(u))\big).$$

Now we write using Proposition 7.16 $H^{2i}(\overline{X}(w)) - H^{2i}_c(\overline{X}(s_l, w_1s^2e_2))(-1) = H^{2i}(\overline{X}(v)) - H^{2i}_c(\overline{X}(t_r, w_1t^2w_2))(-1)$. By the induction hypothesis we have gradings on $H^{2i}(\overline{X}(w))$, $H^{2i}(\overline{X}(u))$ etc. and the identity

$$\mathrm{Im}(g_w) = \ker(f_w) = \bigoplus_{\substack{z \preceq w, s_l | z \\ \ell(z)=i}} H(w)_z = \bigoplus_{\substack{z/s_l \prec w_1tsw_2 \\ \ell(z/s_l)=i-1}} H(w)_z(-1).$$

Here the expression $H^{2i}_c(X(s_l, w_1s^2w_2)) = H^{2i-2}(\overline{X}(w_1sw_2))(-1)$ corresponds to the direct sum $\bigoplus_{\substack{z/s_l \prec w_1s_rw_2 \\ \ell(z/s_l)=i-1}} H(w)_z(-1)$. Now we set $H(v)_{z'} = H(w)_{R(z')}$ for all $z' \preceq w_1tstw_2$ with $z' \notin Q(t_t, w_1t^2w_2)$. Further we choose an arbitrary grading on $H^{2i}_c(X(t_r, w_1t^2w_2)) = H^{2i-2}(X(w_1tw_2))(-1)$. All statements follow for v.

Subcase b). $u = v_1tstw_2$ (or $u = w_1tstv_2$).

Here the result follows by writing f_v as the sum of the homomorphisms

$$H^{2i}(\overline{X}(v)) - H^{2i-2}(\overline{X}(w_1tw_2))(-1) \longrightarrow H^{2i}(\overline{X}(u)) - H^{2i-2}(\overline{X}(v_1tw_2))(-1)$$

and

$$H^{2i-2}(\overline{X}(w_1tw_2))(-1) \longrightarrow H^{2i-2}(\overline{X}(v_1tw_2))(-1).$$

By induction on the length the statements are true for the second homomorphism. Let $r = w_1/v_1$. By induction hypothesis we have gradings on $H^{2i}(\overline{X}(w))$, $H^{2i}(\overline{X}(\bar{u}))$ (where $\overline{u} = v_1 stsw_2$) etc. and the identity

$$\mathrm{Im}(g_w) = \ker(f_w) = \bigoplus_{\substack{z \preceq w, r|z \\ \ell(z)=i}} H(w)_z = \bigoplus_{\substack{z' \prec v_1 stsw_2 \\ \ell(z')=i-1}} H(w)_{z'}(-1).$$

Here $\ker(H^{2i-2}(\overline{X}(w_1sw_2))(-1) \longrightarrow H^{2i-2}(\overline{X}(v_1sw_2))(-1)) = \bigoplus_{\substack{z \preceq w_1sw_2, r|z \\ \ell(z)=i-1}} H(w_1sw_2)_z$ corresponds to $\bigoplus_{\substack{z \preceq w_1s^2w_2, r,s|z \\ \ell(z)=i}} H(w)_z$.

Arguing similar in the same way (but backwards) and using induction with respect to $H^{2i-2}(\overline{X}((w_1tw_2))(-1) \longrightarrow H^{2i-2}(\overline{X}((v_1tw_2))(-1)$ the claim follows.

Subcase c) $u = w_1t^2w_2$.

This case can be avoided by the proceeding lemma. □

Remark 8.2. Of course, it would be interesting to know if Subcase c) holds true anyway.

Let $u, v \in F^+$ with $u \prec v$ with $\ell(u) = \ell(v) - 1$. There is the obvious notion of simultaneous transformation applied to the pair (v, u) with respect to the operations C, K, R, as long as the corresponding subword s, st, sts is part of u, as well. Apart from the critical situation where $v = v_1 stsv_2$ and $u = v_1 ssv_2$, we extend the simultaneous transformation to the tuple (v, u) by letting act C, K, R trivially on u.

Lemma 8.3. *Let $u, v \in F^+$ with $u \prec v$ with $\ell(u) = \ell(v) - 1$. Then there exists a sequence $(v_i, u_i), i = 0, \ldots, n,$ of such tuples such that $(v, u) = (v_0, u_0)$, v_n is of the shape $sv's$ and (v_{i+1}, u_{i+1}) is induced simultaneously from (v_i, u_i) via one of the operations (I) - (III). Here the situation that $(v_i, u_i) = (v_1 stsv_2, v_1 s^2 v_2)$ and $(v_{i+1}, u_{i+1}) = (v_1 tstv_2, v_1 t^2 v_2)$ does not occur.*

Proof. By Theorem 4.1 resp. Lemma 7.1 we may transform v into the desired shape. We only have to analyse the critical case which we want to avoid. After a series of cyclic shifts we may then suppose that $v = tsv's$ and $u = sv's$. If $\gamma(u) \notin W$, then we may transform by Lemma 7.1 u without cyclic shifts into the shape $u_1 r^2 u_2$ for some $r \in S$. But then v can be transformed into the word $tu_1r^2u_2$ and the claim follows using cyclic shifts again. If $\gamma(u) \in W$ but $\gamma(v) \notin W$, then we may write $\gamma(u) = tw'$ for some $w' \in W$ and we are done again.

So it remains to consider the case where $\gamma(v) \in W$. Let $s = s_{i+1}$ and $t = s_i$ (The case $s = s_i$ and $t = s_{i+1}$ behaves symmetrically.).

If we have fixed a reduced decomposition of $\gamma(v)$, then we denote for each $t \in S$ by $m_t \in \mathbb{Z}_{\geq 0}$ its multiplicity in v. Now choose under all reduced expressions (without using the forbidden replacement $sts \rightsquigarrow tst$) and all cyclic shifts (if we get an element which is not reduced, we are done by the first case) of v one such that $(m_{n-1}, m_{n-2}, \ldots, m_{i+2}, m_{i+1})$ is maximal for the lexicographical order. Then let $i+1 \leq j \leq n-1$ be the unique index with with $m_j > m_k$ for all $k > j$ and $m_j \geq m_k$ for all $k \leq j$. We claim that the resulting element z has the shape $s_j z' s_j$. Suppose that the claim is wrong, i.e. $z = z_1 s_j z_2 s_j z_3$ with $z_1 \neq e$ or $z_3 \neq e$ and $s_j \notin \mathrm{supp}(z_1) \cup \mathrm{supp}(z_3)$. Consider two consecutive simple reflections s_j together with the corresponding subword $s_j u' s_j$ of z. If there is no simple reflection s_{j-1} appearing in u' there must be (as $z \in W$) some simple reflection s_{j+1} in between. Hence we can finally replace an expression $s_j s_{j+1} s_j$ in $s_j u s_j$ by $s_{j+1} s_j s_{j+1}$. This gives a contradiction to the maximality of $(m_{n-1}, m_{n-2}, \ldots, m_{i+2}, m_{i+1})$.

Hence there is at least one simple reflection s_{j-1} between two consecutive s_j. Suppose first that $m_{j-1} < m_j$. Thus $s_{j-1} \notin \mathrm{supp}(z_1) \cup \mathrm{supp}(z_3)$. Hence there is some simple reflection s_{j+1} appearing to the left resp. to the right of one of the two exterior s_j. Using a cyclic shift, we can again proceed with a replacement $s_j s_{j+1} s_j \rightsquigarrow s_{j+1} s_j s_{j+1}$ leading to a contradiction with respect to the maximality $(m_{n-1}, m_{n-2}, \ldots, m_{i+2}, m_{i+1})$.

Let $m_j = m_{j-1}$ and suppose that we are in the situation that there is a simple reflection s_{j-1} to the left resp. to the right of one of the two exterior s_j, i.e. $s_{j-1} \in \mathrm{supp}(z_1)$ or $s_{j-1} \in \mathrm{supp}(z_3)$. Then by what we observed above we see that s_{j-1} commutes with any simple reflection appearing in z_1 resp. z_3. Again one verifies that we can apply a cyclic shift and a replacement of the shape $s_j s_{j+1} s_j \rightsquigarrow s_{j+1} s_j s_{j+1}$. Hence we get a contradiction to the maximality of $(m_{n-1}, m_{n-2}, \ldots, m_{i+2}, m_{i+1})$.

□

Example 8.4. a) Let $(1,5) = (1,2)(2,3)(3,4)(4,5)(3,4)(2,3)(1,2) = sw's$. Then $sw' = (1,2)(2,3)(3,4)(4,5)(3,4)(2,3)$. Shifting $(2,3)$ from the RHS to the LHS we get from (sw', w') the tuple $((2,3)(1,2)(2,3)(3,4)(4,5)(3,4), (2,3)(2,3)(3,4)(4,5)(3,4))$ which we want to avoid. Instead we consider the sequence

$$\begin{aligned}(v_1, u_1) &= ((1,2)(2,3)(4,5)(3,4)(4,5)(2,3), (2,3)(4,5)(3,4)(4,5)(2,3)) \\ &\vdots \\ (v_4, u_4) &= ((4,5)(1,2)(2,3)(3,4)(2,3)(4,5), (4,5)(2,3)(3,4)(2,3)(4,5)).\end{aligned}$$

b) Let $sw' = (3,4)(2,3)(1,2)(3,4)(4,5)(2,3)$. Again shifting the reflection $(2,3)$ from the RHS to the LHS, would yield a not desirable tuple. Instead we consider the sequence

$$\begin{aligned}(v_1, u_1) &= ((3,4)(2,3)(3,4)(1,2)(4,5)(2,3), (2,3)(3,4)(1,2)(4,5)(2,3)),\\ (v_2, u_2) &= ((2,3)(3,4)(2,3)(1,2)(4,5)(2,3), (2,3)(3,4)(1,2)(4,5)(2,3)).\end{aligned}$$

Now we come to the final aspect of the introduction.

Conjecture 8.5. *Let $w \in F^+$. Then for $v \preceq w$, there are gradings $H^{2i}(\overline{X}(v)) = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} H(v)_z$ such that the complex*

$$0 \longrightarrow H^{2i}(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(v)) \longrightarrow \cdots \longrightarrow H^{2i}(\overline{X}(e)) \tag{8.2}$$

is quasi-isomorphic to the graded direct sum of complexes of the shape

$$0 \to i^G_{P^w_z} \to \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-1}} i^G_{P^v_z} \to \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-2}} i^G_{P^v_z} \to \cdots \to i^G_{P^e_z} \to 0.$$

as in section 1, cf. (2.8).

By Proposition 3.9 the conjecture is true for Coxeter elements. Thus we deal in what follows with elements of positive height.

Proposition 8.6. *Let $w \in F^+$ with $\mathrm{ht}(w) \geq 1$. Then for proving the conjecture we may assume that w is of the form $w = sw's$.*

Proof. We apply again the operations (I) - (III) to the complex. Suppose that the statement is true for w. We need to show that the assertion is true for the transformed element in F^+.

(I) Let $w = sw'$. Then we set $H(C(v))_{C(z)} = H(v)_z$ for $v \preceq w$ and $z \preceq v$. The resulting complexes are clearly quasi-isomorphic.

(II) Let $w = w_1 st w_2$ with $st = ts$. We set $H(K(v))_{K(z)} = H(v)_z$ for $v \preceq w$ and $z \preceq v$. The resulting complexes are clearly quasi-isomorphic.

(III) Let $w = w_1 sts w_2$ and $R(w) = w_1 tst w_2$. By Proposition 7.16 we know that for $z_i \preceq w_i$, $i = 1, 2$, the cohomology $H^{2i}_c(X(s_l, z_1 sts z_2))$ is a direct factor of $H^{2i}(\overline{X}(z_1 sts z_2))$ and that

$$H^{2i}(\overline{X}(z_1 sts z_2)) - H^{2i}_c(X(s_l, z_1 s^2 z_2)) = H^{2i}(\overline{X}(z_1 tst z_2)) - H^{2i}_c(X(t_r, z_1 t^2 z_2)).$$

Further $H^{2i}(\overline{X}(z_1 s^2 z_2)) = H^{2i}(\overline{X}(z_1 s z_2)) \bigoplus H^{2i-2}(\overline{X}(z_1 s z_2))(-1)$. Hence the complex (8.2) for w is quasi-isomorphic to

$$\begin{tikzcd}
 & \bigoplus\limits_{\substack{v_1 \prec w_1 \\ \ell(v_1)=\ell(w_1)-1}} H^{2i}(\overline{X}(v_1stsw_2)) - H_c^{2i}(X(s_l, v_1s^2w_2)) & \\
H^{2i}(\overline{X}(w)) - H_c^{2i}(X(s_l, w_1s^2w_2)) \arrow[ur] \arrow[r] \arrow[dr] & H^{2i}(\overline{X}(w_1stw_2)) \oplus H^{2i}(\overline{X}(w_1tsw_2)) \arrow[r] & \cdots \\
 & \bigoplus\limits_{\substack{v_2 \prec w_2 \\ \ell(v_2)=\ell(w_2)-1}} H^{2i}(\overline{X}(w_1stsv_2)) - H_c^{2i}(X(s_l, w_1s^2v_2)) &
\end{tikzcd}$$

which coincides with

$$\begin{tikzcd}
 & \bigoplus\limits_{\substack{v_1 \prec w_1 \\ \ell(v_1)=\ell(w_1)-1}} H^{2i}(\overline{X}(v_1tstw_2)) - H_c^{2i}(X(t_r, v_1t^2w_2)) & \\
H^{2i}(\overline{X}(R(w))) - H_c^{2i}(X(t_r, w_1t^2w_2)) \arrow[ur] \arrow[r] \arrow[dr] & H^{2i}(\overline{X}(w_1stw_2)) \oplus H^{2i}(\overline{X}(w_1tsw_2)) \arrow[r] & \cdots \\
 & \bigoplus\limits_{\substack{v_2 \prec w_2 \\ \ell(v_2)=\ell(w_2)-1}} H^{2i}(\overline{X}(w_1tstv_2)) - H_c^{2i}(X(t_r, w_1t^2v_2)) &
\end{tikzcd}$$

By reversing the above argument, i.e. by adding the contributions $H_c^{2i}(X(t_r, z_1t^2z_2))$ and $H^{2i-2}(X(z_1tz_2))(-1)$ to the complex (with the obvious gradings) we see that the transformed element $R(w)$ satisfies the claim, as well. $\square$

Remark 8.7. Let $w = sw's$. For every $v' \preceq w'$, we have the identity $H^{2i}(\overline{X}(sv's)) = H^{2i}(\overline{X}(v's)) \bigoplus H^{2i-2}(\overline{X}(v's))(-1)$. Hence the complex (8.2) is quasi-isomorphic to

(8.3)
$$\begin{tikzcd}
 & H^{2i}(\overline{X}(sw')) \arrow[r] & \bigoplus\limits_{\substack{v' \prec w' \\ \ell(v')=\ell(w')-1}} H^{2i}(\overline{X}(sv')) \bigoplus H^{2i}(X(w')) & \\
 & & & \cdots \\
H^{2i-2}(\overline{X}(w's))(-1) \arrow[uur] \arrow[rr] & & \bigoplus\limits_{\substack{v' \prec w' \\ \ell(v')=\ell(w')-1}} H^{2i-2}(\overline{X}(v's))(-1). \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Moreover, by considering the middle column in the diagram (8.1), we see that the complex

$$H_c^{2i}(X(s, sw's)) \longrightarrow \bigoplus_{\substack{v' \prec w' \\ \ell(v')=\ell(w')-1}} H_c^{2i}(X(s, sv's)) \longrightarrow \cdots \longrightarrow H_c^{2i}(X(s, ss))$$

is the direct sum of the complexes

$$H_c^{2i}(X(s^2, sw's))\langle -i\rangle \longrightarrow \bigoplus_{\substack{v'\prec w'\\ \ell(v')=\ell(w')-1}} H_c^{2i}(X(s^2, sv's))\langle -i\rangle \longrightarrow \cdots \longrightarrow H_c^{2i}(X(s^2))\langle -i\rangle$$

and

$$H_c^{2i}(X(s, sw')) \longrightarrow \bigoplus_{\substack{v'\prec w'\\ \ell(v')=\ell(w')-1}} H_c^{2i}(X(s, sv')) \longrightarrow \cdots \longrightarrow H_c^{2i}(X(s)).$$

Theorem 8.8. *Let $w \in F^+$. Then the conjecture is true for $i = 0, 1, \ell(w)-1, \ell(w)$.*

Proof. We may suppose that w has full support. If $\mathrm{ht}(w) = 0$, then the claim follows from Proposition 3.9. So we assume in the sequel that $\mathrm{ht}(w) \geq 1$.

If $i = 0$ then the complex coincides with the complex (2.6) of section 1 which yields the Steinberg representation v_B^G.

If $i = \ell(w)$, the claim is trivial.

If $i = \ell(w) - 1$ the assertion follows from Corollary 7.9.

So let $i = 1$. The proof is by induction on the length. By Proposition 8.6 we may assume that $w = sw's$. For the start of induction let $w = sw's$ with $s = s_i$ and

$$w' = s_1 \cdots s_{i-1}s_{i+1} \cdots s_{n-1}.$$

Then the lower line in the complex (8.3) is nothing else but the complex (2.7). In view of Lemma 8.1 we define a grading on the upper line as follows. Set for all $j \neq i$,

$$H(ss_j)_s \quad = \quad H(ss_j)_e(-1) = i^G_{P(ss_j)}(-1)$$

and

$$H(t)_t = H(t)_e(-1) = i^G_{P(t)}(-1) \quad \forall t \in S.$$

In particular, we have fixed $H(ss_j)_{s_j}$ in this way for all $j \neq i$. Further we set for $j < i-1$,

$$H(s_js_{j+1})_{s_j} = i^G_{P(s_j)}(-1)$$

and

$$H(s_js_{j+1})_{s_{j+1}} = i^G_{P(s_{j+1})}(-1)$$

for $j > i$.

If $s_js_k = s_ks_j$, then there is a canonical grading on $H^2(\overline{X}(s_js_k))$. Thus we have defined gradings for all subwords of sw' of length ≤ 2. Now we extend the above gradings to the complex

$$0 \longrightarrow H^2(\overline{X}(w')) \longrightarrow \bigoplus_{\substack{v'\prec w'\\ \ell(v')=\ell(w')-1}} H^2(\overline{X}(v')) \longrightarrow \cdots \longrightarrow H^2(\overline{X}(e))$$

inductively in the following way. For $v' \preceq w'$ and $z \prec v'$, set $H(v')_z = \bigcap_{\substack{u' \prec v' \\ \ell(u')=\ell(v')-1}} H(u')_z$. Finally we apply Lemma 8.1 once again in order to get the remaining gradings on the upper line, and hence on the whole complex which satisfies the claim. Moreover, we see that the differential map

$$\bigoplus_{\substack{v \preceq w \\ \ell(v)=2}} H^2(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{t \prec w \\ \ell(t)=1}} H^2(\overline{X}(t)) \tag{8.4}$$

is surjective.

Let's proceed with the induction step. So let $w = sw's \in F^+$ with $\mathrm{ht}(sw') \geq 1$.

Claim: The map (8.4) is surjective, as well.

In fact we consider the complex (8.3). By induction hypothesis we deduce that the map $\bigoplus_{\substack{v \preceq sw' \\ \ell(v)=2}} H^2(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{t \prec sw' \\ \ell(t)=1}} H^2(\overline{X}(t))$ is surjective. On the other hand, we have a surjection $H^2(\overline{X}(s^2)) \longrightarrow H^2(\overline{X}(s_r))$. The claim follows.

We distinguish finally the following cases.

Case a). $s \in \mathrm{supp}(w')$. Then the lower line in the complex (8.3) coincides with the complex (2.7). It is contractible by Proposition 2.12. By induction hypothesis the statement is true for the upper line. The grading is defined by $H(w)_t := H(sw')_t$ for $t \preceq sw'$ resp. $H(w)_{s_r} := H(w's)_e$.

Case b). $s \notin \mathrm{supp}(w')$. Then the lower line in the complex (8.3) coincides with the complex (2.7) and gives a resolution of the generalized Steinberg representation $v^G_{P(s)}$. By induction hypothesis the statement is true for the upper line. As the map $\bigoplus_{\substack{v \preceq sw' \\ \ell(v)=2}} H^2(\overline{X}(v)) \longrightarrow \bigoplus_{\substack{t \prec sw' \\ \ell(t)=1}} H^2(\overline{X}(t))$ is surjective the representation $v^G_{P(s)}$ occurs in the cohomology of $X(w)$. The grading is defined by $H(w)_t := H(sw')_t$ for $t \preceq sw'$ resp. $H(w)_{s_r} := H(w's)_e$. □

By the proof of the preceding theorem we get an inductively formula for the Tate twist -1 contribution of the cohomology of DL-varieties.

Corollary 8.9. *Let $w = sw's \in F^+$ with $\mathrm{ht}(sw') \geq 1$. Then*

$$H^*_c(X(w))\langle -1 \rangle = \begin{cases} H^*_c(X(sw'))\langle -1 \rangle[-1] & \text{if } s \in \mathrm{supp}(w') \\ H^*_c(X(sw'))\langle -1 \rangle[-1] \bigoplus v^G_{P(s)}(-1)[-\ell(w)] & \text{if } s \notin \mathrm{supp}(w') \end{cases}.$$

In view of the lower line in (8.3) we generalize Conjecture 3.10.

Conjecture 8.10. *Let $w \in F^+$ and fix $u \prec w$. For $u \preceq v \preceq w$, there are gradings $H^{2i}(\overline{X}(v)) = \bigoplus_{\substack{z \preceq v \\ \ell(z)=i}} H(v)_z$ such that the complex*

$$0 \longrightarrow H^{2i}(\overline{X}(w)) \longrightarrow \bigoplus_{\substack{u \preceq v \prec w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(v)) \longrightarrow \cdots \longrightarrow H^{2i}(\overline{X}(u)) \tag{8.5}$$

is quasi-isomorphic to the graded direct sum of complexes of the shape

$$0 \to i^G_{P^w_z} \to \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-1}} i^G_{P^v_z} \to \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-2}} i^G_{P^v_z} \to \cdots \to i^G_{P^e_z} \to 0$$

cf. (2.8).

Remark 8.11. As in Proposition 8.6 one can reduce the conjecture to the case where w is of the shape $w = sw's$. Again we only have seriously to consider the operation (III) in this process of transformations. So let $w = w_1stsw_2$ and $R(w) = \overline{w} = w_1tstw_2$. If

$$u = v_1stsv_2, v_1s^2v_2, v_1stv_2, v_1tsv_2, v_1v_2,$$

respectively, we set

$$\overline{u} = v_1tstv_2, v_1t^2v_2, v_1tsv_2, v_1stv_2, v_1v_2.$$

Then the complex (8.5) is quasi-isomorphic to

$$0 \longrightarrow H^{2i}(\overline{X}(\overline{w})) \longrightarrow \bigoplus_{\substack{u \preceq v \prec w \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(\overline{v})) \longrightarrow \cdots \longrightarrow H^{2i}(\overline{X}(\overline{u})). \tag{8.6}$$

Similarly, we transform the lower complex in the conjecture above.

Remark 8.12. If the conjecture is true then for $w = sw's$ the gradings considered on $H^{2i}(\overline{X}(sw'))$ in the upper line in (8.6) are not necessarily in the way that the induced complex is quasi-isomorphic to the complex

$$0 \longrightarrow H^{2i}(\overline{X}(sw')) \longrightarrow \bigoplus_{\substack{v \prec sw' \\ \ell(v)=\ell(w)-1}} H^{2i}(\overline{X}(v)) \longrightarrow \cdots \longrightarrow H^{2i}(\overline{X}(e)).$$

Remark 8.13. Suppose that Conjecture 8.2 is true. Then we can reprove the statement in Remark 5.11 concerning the appearing of the trivial representation in the cohomology of a DL-variety $X(w)$. Indeed, the multiplicity of the trivial representation i^G_G in an induced representation i^G_P is always 1. Hence for any $z \preceq w$, the v-contribution

$$0 \to i^G_{P^w_z} \to \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-1}} i^G_{P^v_z} \to \bigoplus_{\substack{z \preceq v \preceq w \\ \ell(v)=\ell(w)-2}} i^G_{P^v_z} \to \cdots \to i^G_{P^e_z} \to 0$$

restricted to i^G_G is acyclic as the resulting index set is contractible (a lattice). It follows that the i^G_G occurs only in the top cohomology group of $X(w)$.

9. Examples

Here we present some examples concerning Theorem 8.8. In the following we omit the Tate twists for reasons of clarity. For $w \in F^+$ and $z \preceq w$, we write $i^G_P(z)$ instead of $H(w)_z = i^G_P$.

a) Let $G = \mathrm{GL}_3$ and let $w = (1,2)(2,3)(1,2) \in F^+$.

Let $i = 2$. Here the complex (8.3) is

$$\begin{tikzcd} & H^4(\overline{X}((1,2)(2,3))) = i^G_G \\ H^2(\overline{X}((2,3)(1,2)) \arrow[r] \arrow[ur] & H^2(\overline{X}((1,2)_r)) = i^G_{P_{(2,1)}}. \end{tikzcd}$$

We consider the grading $H^2(\overline{X}(2,3)(1,2)) = i^G_G(2,3) \bigoplus i^G_{P_{(2,1)}}(1,2)$. Thus we see that the above complex is contractible, as it should be by Example 4.15.

Let $i = 1$. Here the complex (8.3) is

$$\begin{tikzcd} H^2(\overline{X}((1,2)(2,3))) \arrow[r] & H^2(\overline{X}((1,2))) \bigoplus H^2(X((2,3))) = i^G_{P_{(2,1)}} \bigoplus i^G_{P_{(2,1)}} \\ H^0(\overline{X}((2,3)(1,2))) = i^G_G \arrow[r] \arrow[u, "\nearrow"'] & H^2(\overline{X}((1,2)_r)) = i^G_{P_{(2,1)}}. \arrow[u, "\nearrow"'] \end{tikzcd}$$

We consider the grading $H^2(\overline{X}((2,3)(1,2))) = i^G_G(1,2) \bigoplus i^G_{P_{(2,1)}}(2,3)$. Thus we see that the above complex is contractible, as it should be by Example 4.15.

b) Let $G = \mathrm{GL}_4$ and let $w = (3,4)(1,2)(2,3)(3,4) \in F^+$.

Let $i = 2$. We have $H^4(\overline{X}((1,2)(2,3))) = i^G_{P_{(3,1)}}$. We consider the grading

$$H^4(\overline{X}((3,4)(1,2)(2,3)) = i^G_G((3,4)(2,3)) \bigoplus i^G_{P_{(3,1)}}((1,2)(2,3)) \bigoplus i^G_{P_{(2,2)}}((1,2)(3,4)).$$

The reduced complex (8.3) is given by

$$\begin{tikzcd} & H^4(\overline{X}((1,2)(2,3))) = i^G_{P_{(3,1)}} \\ H^4(\overline{X}((3,4)(1,2)(2,3)))) \arrow[ur] \arrow[r] \arrow[dr] & H^4(\overline{X}((3,4)(1,2))) = i^G_{P_{(2,2)}} \\ & H^4(\overline{X}((3,4)(2,3))) = i^G_{P_{(1,3)}} \end{tikzcd}$$

$$
\begin{array}{ccccc}
 & & \uparrow & & \\
 & & H^2(\overline{X}((1,2)(3,4))) & & \\
\uparrow & \nearrow & & \searrow & \\
H^2(\overline{X}(\mathrm{Cox})) & & \uparrow & & H^2(\overline{X}((3,4))) = i^G_{P_{(1,1,2)}}. \\
 & \searrow & & \nearrow & \\
 & & H^2(\overline{X}((2,3)(3,4))) & &
\end{array}
$$

We consider the gradings

$$
\begin{aligned}
H^2(\overline{X}((2,3)(3,4))) &= i^G_{P_{(1,3)}}(2,3) \bigoplus i^G_{P_{(1,1,2)}}(3,4), \\
H^2(\overline{X}((1,2)(3,4))) &= i^G_{P_{(2,2)}}(1,2) \bigoplus i^G_{P_{(2,2)}}(3,4)
\end{aligned}
$$

and

$$H^2(\overline{X}(\mathrm{Cox})) = i^G_G(2,3) \bigoplus i^G_{P_{(3,1)}}(3,4) \bigoplus i^G_{P_{(2,2)}}(1,2).$$

We get $H^*_c(X(w))\langle -2\rangle = j_{(2,2)}[-5]$, as it should be by Example 5.12.

Let $i = 1$. We consider for $w' = (1,2)(2,3)$ the graded complex

$$
\begin{array}{ccc}
 & & H^2(\overline{X}((1,2))) = i^G_{P_{(2,1,1)}} \\
 & \nearrow & \\
H^2(\overline{X}((1,2)(2,3))) & & \\
 & \searrow & \\
 & & H^2(\overline{X}((2,3))) = i^G_{P_{(1,2,1)}}
\end{array}
$$

with $H^2(\overline{X}((1,2)(2,3))) = i^G_{P_{(3,1)}}(2,3) \bigoplus i^G_{P_{(2,1,1)}}(1,2)$.

We consider further the gradings

$$
\begin{aligned}
H^2(\overline{X}((3,4)(2,3))) &= i^G_{P_{(3,1)}}(3,4) \bigoplus i^G_{P_{(1,2,1)}}(2,3), \\
H^2(\overline{X}((1,2)(3,4))) &= i^G_{P_{(2,2)}}(1,2) \bigoplus i^G_{P_{(2,2)}}(3,4)
\end{aligned}
$$

and

$$H^2(\overline{X}((3,4)(1,2)(2,3))) = i^G_G(3,4) \bigoplus i^G_{P_{(3,1)}}(2,3) \bigoplus i^G_{P_{(2,2)}}(1,2).$$

The reduced complex (8.3) is given by

$$\begin{array}{ccccc}
 & & H^2(\overline{X}((1,2)(2,3))) & \longrightarrow & H^2(X((1,2)) = i^G_{P_{(2,1,1)}} \\
 & \nearrow & & & \\
H^2(\overline{X}((3,4)(1,2)(2,3))) & \longrightarrow & H^2(\overline{X}((3,4)(1,2))) & \longrightarrow & H^2(\overline{X}((2,3))) = i^G_{P_{(1,2,1)}} \\
 & \searrow & & & \\
 & & H^2(\overline{X}((3,4)(2,3))) & \longrightarrow & H^2(\overline{X}((3,4))) = i^G_{P_{(1,1,2)}}
\end{array}$$

$$\begin{array}{ccccc}
 & & \uparrow & & \\
 & & H^0(\overline{X}((1,2)(3,4))) = i^G_{P_{(2,2)}} & & \\
\uparrow & \nearrow & & \searrow & \uparrow \\
H^0(\overline{X}(\mathrm{Cox})) = i^G_G & & \uparrow & & H^0(\overline{X}((3,4))) = i^G_{P_{(1,1,2)}}. \\
 & \searrow & & \nearrow & \\
 & & H^0(\overline{X}((2,3)(3,4))) = i^G_{P_{(1,3)}} & &
\end{array}$$

It follows that the complex is contractible, as it should be by Example 5.12.

c) Let $G = \mathrm{GL}_4$ and let $w = (2,3)(1,2)(3,4)(2,3) \in F^+$.

Let $i = 2$. We have $H^4(\overline{X}((1,2)(3,4))) = i^G_{P_{(2,2)}}$. We consider the grading

$$H^4(\overline{X}((2,3)(1,2)(3,4)) = i^G_G((2,3)(1,2)) \bigoplus i^G_{P_{(3,1)}}((2,3)(3,4)) \bigoplus i^G_{P_{(2,2)}}((1,2)(3,4)).$$

The reduced complex (8.3) is given by

$$\begin{array}{ccc}
 & & H^4(\overline{X}((1,2)(3,4))) = i^G_{P_{(2,2)}} \\
 & \nearrow & \\
H^4(\overline{X}((2,3)(1,2)(3,4)))) & \longrightarrow & H^4(\overline{X}((2,3)(1,2))) = i^G_{P_{(3,1)}} \\
 & \searrow & \\
 & & H^4(\overline{X}((2,3)(3,4))) = i^G_{P_{(1,3)}}
\end{array}$$

$$\begin{array}{ccccc}
 & & \uparrow & & \\
 & & H^2(\overline{X}((1,2)(2,3))) & & \\
\uparrow & \nearrow & & \searrow & \\
H^2(\overline{X}((1,2)(3,4)(2,3))) & & \uparrow & & H^2(\overline{X}((2,3))) = i^G_{P_{(1,2,1)}}. \\
 & \searrow & & \nearrow & \\
 & & H^2(\overline{X}((3,4)(2,3))) & &
\end{array}$$

We consider the gradings

$$
\begin{aligned}
H^2(\overline{X}((3,4)(2,3))) &= i^G_{P_{(1,3)}}(3,4) \bigoplus i^G_{P_{(1,2,1)}}(2,3), \\
H^2(\overline{X}((1,2)(2,3))) &= i^G_{P_{(3,1)}}(1,2) \bigoplus i^G_{P_{(1,2,1)}}(2,3)
\end{aligned}
$$

and

$$
H^2(\overline{X}((1,2)(3,4)(2,3))) = i^G_G(1,2) \bigoplus i^G_{P_{(3,1)}}(3,4) \bigoplus i^G_{P_{(2,2)}}(2,3).
$$

We get $H^*_c(X(w))\langle -2\rangle = i^G_{P_{(2,1,1)}}/i^G_{P_{(2,2)}}[-5]$, as it should be by Example 5.13.

Let $i = 1$. We consider for $w' = (1,2)(3,4)$ the graded complex

$$
\begin{array}{ccc}
 & & H^2(\overline{X}((1,2))) = i^G_{P_{(2,1,1)}} \\
 & \nearrow & \\
H^2(\overline{X}((1,2)(3,4))) & & \\
 & \searrow & \\
 & & H^2(\overline{X}((3,4))) = i^G_{P_{(1,1,2)}}
\end{array}
$$

with $H^2(\overline{X}((1,2)(3,4))) = i^G_{P_{(2,2)}}(1,2) \bigoplus i^G_{P_{(2,2)}}(3,4)$. We consider further the gradings

$$
\begin{aligned}
H^2(\overline{X}((3,4)(2,3))) &= i^G_{P_{(1,3)}}(2,3) \bigoplus i^G_{P_{(1,1,2)}}(3,4), \\
H^2(\overline{X}((1,2)(2,3))) &= i^G_{P_{(3,1)}}(2,3) \bigoplus i^G_{P_{(2,1,1)}}(1,2)
\end{aligned}
$$

and

$$
H^2(\overline{X}((1,2)(3,4)(2,3))) = i^G_G(2,3) \bigoplus i^G_{P_{(3,1)}}(3,4) \bigoplus i^G_{P_{(2,2)}}(1,2).
$$

The reduced complex (8.3) is given by

$$
\begin{array}{ccccc}
 & & H^2(\overline{X}((2,3)(1,2))) & \longrightarrow & H^2(X((1,2))) = i^G_{P_{(2,1,1)}} \\
 & \nearrow & & & \\
H^2(\overline{X}((2,3)(1,2)(3,4))) & \longrightarrow & H^2(\overline{X}(((1,2)(3,4))) & \longrightarrow & H^2(\overline{X}((2,3))) = i^G_{P_{(1,2,1)}} \\
 & \searrow & & & \\
 & & H^2(\overline{X}((2,3)(3,4))) & \longrightarrow & H^2(\overline{X}((3,4))) = i^G_{P_{(1,1,2)}}
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \uparrow & & \\
 & & H^0(\overline{X}((2,3)(1,2))) = i^G_{P_{(3,1)}} & & \\
\uparrow & \nearrow & & \searrow & \uparrow \\
H^0(\overline{X}((2,3)(1,2)(3,4))) = i^G_G & & \uparrow & & H^0(\overline{X}((2,3))) = i^G_{P_{(1,2,1)}}. \\
 & \searrow & & \nearrow & \\
 & & H^0(\overline{X}((3,4)(2,3))) = i^G_{P_{(1,3)}} & &
\end{array}
$$

We get $H^*_c(X(w))\langle -1\rangle = j_{(2,2)}[-4]$, as it should be by Example 5.13.

10. Appendix A

In this section we present a different but more vague method for determining the cohomology of DL-varieties attached to elements in the Weyl group. Here the approach is the other way round compared to the short version. Some results presented here might be also treated by using Demazure resolutions and the results of the previous sections.

Let $w = sw's \in W$ with $\ell(w) = \ell(w') + 2$ and $Z = X(w) \cup X(sw') \subset X$ as before. Reconsider for $i \geq 0$ the natural map $r^i = r^i_{w,sw'} : H^i_c(Z) \longrightarrow H^i_c(X(sw'))$. Now write $H^i_c(Z) = H^{i-2}_c(Z')(-1) = A \bigoplus B$ where

$$A \cong \operatorname{coker}\big(H^{i-3}_c(X(w')) \longrightarrow H^{i-2}_c(X(w's))\big)(-1)$$

and

$$B = \ker\big(H^{i-2}_c(X(w')) \longrightarrow H^{i-1}_c(X(w's))\big)(-1).$$

By Remark 4.13 we know that $r^i_{|A} = 0$.

Motivated by Proposition 5.16 we pose the following conjecture.

Conjecture 10.1. *For $i \geq 0$, the map $r^i_{|B} : B \longrightarrow H^i_c(X(sw'))$ has si-full rang.*

Remark 10.2. For our purpose it is even enough to have the validity of a weaker form of the above conjecture. More precisely, it suffices to know that for all $V \in \operatorname{supp}(H)$ with $i = 2t(V)$ the map $r^i_{|B^V} : B^V \longrightarrow H^{2i}_c(X(sw'))^V$ has full rang.

Indeed if the assumption $i = 2t(V)$ is not satisfied, then we proceed as follows to determine the V-isotypic part of the map $r^i_{w,sw'}$. We fix a reduced decomposition of w'. Since $i > 2t(V)$ there is by purity of $\overline{X}(w)$ some hypersquare $Q \subset F^+$ of dimension d (which is assumed to be minimal) with head w and with $\{w, sw'\} \subset Q$ and such that $V \subset H^i_c(X(w) \cup X(sw'))$ is induced by $H^{i-1}_c(X(Q) \setminus X(w) \cup X(sw'))$ via the corresponding

boundary map. A minimal hypersquare exists since the extreme case where $\mathrm{tail}(Q) = e$ yields one. Then $\mathrm{tail}(Q) = sv'$ or $\mathrm{tail}(Q) = v'$ for some $v' \in F^+$ with $v' \preceq w'$.

Suppose that $d = 2$.

1. Case. $\mathrm{tail}(Q) = sv'$. Then $V \subset H_c^{i-1}(X(sv's) \cup X(sv'))$ maps onto $V \subset H_c^i(X(w) \cup X(sw'))$ via the boundary map.

Subcase a) V is induced by $H_c^{i-1}(X(sv'))$.

Subsubcase i) $V \subset H_c^{i-1}(X(sv'))$ maps to $V \subset H_c^i(X(sw')$ via the boundary map δ^{i-1} : $H_c^{i-1}(X(sv')) \longrightarrow H_c^i(X(sw'))$ (which is known by induction, cf. the following pages). In this case $r^i_{w,sw'}$ maps $V \subset H_c^i(X(w) \cup X(sw'))$ onto $V \subset H_c^i(X(sw'))$ by considering the commutative diagram:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots \longrightarrow & & H_c^i(X(w)) & \longrightarrow & H_c^i(X(w) \cup X(sw')) & \longrightarrow & H_c^i(X(sw')) & & \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots \longrightarrow & & H_c^{i-1}(X(v)) & \longrightarrow & H_c^{i-1}(X(v) \cup X(sv')) & \longrightarrow & H_c^{i-1}(X(sv')) & & \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Subsubcase ii) $V \subset H_c^{i-1}(X(sv'))$ does not map onto $V \subset H_c^i(X(sw')$. In this case $r^i_{w,sw'}$ does not map $V \subset H_c^i(X(w) \cup X(sw'))$ onto $V \subset H_c^i(X(sw'))$ by considering again the above diagram.

Subcase b) V is induced by $H_c^{i-1}(X(v))$. In this case $r^i_{w,sw'}$ does not map $V \subset H_c^i(X(w) \cup X(sw'))$ onto $V \subset H_c^i(X(sw'))$ by considering the above diagram.

2. Case. $\mathrm{tail}(Q) = w'$. This case yields a contradiction since the boundary map $H_c^{i-1}(X(w's) \cup X(w')) \longrightarrow H_c^i(X(w) \cup X(sw'))$ vanishes by Corollary 4.11.

Suppose that $d = 3$.

1. Case $\mathrm{tail}(Q) = sv'$ for some $v' \prec w'$ with $\ell(w') = \ell(v') + 1$, i.e. Q has the shape

$$
Q: \quad \begin{array}{ccccc}
 & & w & & \\
 & \nearrow & \uparrow & \nwarrow & \\
sw' & & sv_1's & & sv_2's \\
\uparrow & \nearrow\nwarrow & & \nearrow\nwarrow & \uparrow \\
sv_1' & & sv_2' & & sv's \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & sv' & &
\end{array}
$$

for some $v_1', v_2' \in F^+$. We set $A = \{sv_1', sv_2', sv'\}$ and $= \{sv_1's, sv_2's, sv's\}$. Then $X(A)$ is closed in $X(Q) \setminus (X(w) \cup X(sw'))$ whereas $X(B)$ is open in the latter space. Now we may imitate the procedure of the case $d = 2$. The variety $X(A)$ corresponds to $X(sv')$ whereas $X(B)$ plays the role of $X(v)$.

2. Case $\mathrm{tail}(Q) = v'$ for some $v' \prec w'$ with $\ell(w') = \ell(v') + 1$, i.e., Q has the shape

$$
Q: \quad \begin{array}{ccccc}
 & & w & & \\
 & \nearrow & \uparrow & \nwarrow & \\
sw' & & sv's & & w's \\
\uparrow & \nearrow\nwarrow & & \nearrow\nwarrow & \uparrow \\
sv' & & w' & & v's \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & v' & &
\end{array}
$$

We claim that $r^i_{w,sw'}$ is trivial. Indeed, as $X(sv's) \cup X(sv')$ and $X(w's) \cup X(w')$ are both open in $X(Q) \setminus (X(w) \cup X(sw'))$ whereas $X(v's) \cup X(v')$ is closed we see that

- $V \subset H_c^{i-1}(X(sv's) \cup X(sv'))$ gives a contradiction to the minimality with respect to d.
- $V \subset H_c^{i-1}(X(w's) \cup X(w'))$ gives a contradiction as the boundary map is trivial.

So $V \subset H_c^{i-1}(X(v's) \cup X(v'))$. But $V \subset H_c^{i-1}(X(Q(v', v)))$ is mapped to $H_c^i(X(Q(w', w)))$ via the boundary map. But the latter one is given by the diagram

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
H_c^i(X(Q(w',w)) & = & H_c^i(X(w) \cup X(sw')) & \oplus & H_c^i(X(w's) \cup X(w')) \\
\uparrow & & \uparrow & & \uparrow \\
H_c^{i-1}(X(Q(v',v)) & = & H_c^{i-1}(X(v) \cup X(sv')) & \oplus & H_c^{i-1}(X(v's) \cup X(v')) \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots
\end{array}
$$

i.e. it is for trivial reasons the direct sum of the summands. Hence we get a contradiction.

The higher dimensional cases $d \geq 4$ behave as above whether $\mathrm{tail}(Q) = sv'$ or $\mathrm{tail}(Q) = v'$. The latter case gives a contradiction. $\square$

Thus for determining the cohomology of $X(w)$, it remains to compute the cohomology of the edge $X(w') \cup X(w's)$ which we explain now.

Let $w, v \in W$ with $\ell(w) = \ell(v) + 1$. We want to determine the cohomology of the locally closed subvariety $X(w) \cup X(v) \subset X$. Suppose that we may write $w = sw's$ as in the previous sections. If $v \in \{sw', w's\}$ then $H^i_c(X(w) \cup X(v)) = H^{i-2}_c(X(w') \cup X(w's))(-1)$ and we may suppose by induction on the length of w that these groups are known.

So let $v = sv's$ with $v' < w'$. Then the cohomology of $X(w) \cup X(v)$ sits in a long exact cohomology sequence

$$\cdots \longrightarrow H^i_c(X(w) \cup X(v)) \longrightarrow H^i_c(X(Q)) \longrightarrow H^i_c(X(sw') \cup X(sv')) \longrightarrow \cdots$$

where Q is the square $Q = \{w, v, sw', sv'\} \subset W$. The cohomology of $X(sw') \cup X(sv')$ is known by induction on the length of $w's$. On the other hand, the square

$$Q: \quad \begin{tikzcd} & w & \\ v \arrow[ur] & & sw' \arrow[ul] \\ & sv'. \arrow[ul] \arrow[ur] & \end{tikzcd}$$

is induced by the square

$$\hat{Q}: \quad \begin{tikzcd} & w's & \\ v's \arrow[ur] & & w' \arrow[ul] \\ & v' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

via multiplication with $s \in S$ from the left. The union $Q \cup \hat{Q}$ gives rise to a cube or a 3-dimensional hypersquare

$$Q \cup \hat{Q}: \quad \begin{tikzcd} & w & \\ sw' \arrow[ur] & v \arrow[u] & sw' \arrow[ul] \\ sv' \arrow[u] \arrow[ur] & w' \arrow[ul] \arrow[ur] & v's \arrow[u] \arrow[ul] \\ & v' \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

Then by Proposition 6.5 the variety $X(Q)$ is an $\mathbb{A}^1$-bundle over $X(\hat{Q})$ and $X(Q \cup \hat{Q})$ is a $\mathbb{P}^1$-bundle over $X(\hat{Q})$. Further the restriction map in cohomology

$$r^i_{Q,\{sw',sv'\}} : H^i_c(X(Q)) \longrightarrow H^i_c(X(sw') \cup X(sv'))$$

can be computed in the same way as in Conjecture 10.1 resp. Remark 10.2. Thus if we are able to determine the cohomology group $H^*_c(X(\hat{Q}))$ we have knowledge of the cohomology of $X(w) \cup X(v)$. Hence we have reduced the question of determining the cohomology of the edge $X(w) \cup X(v)$ by the prize of enlarging the square but where the head has smaller length.

Remark 10.3. In general we have to apply to $w \in W$ the operations (I) - (III) in the previous section in order to write it in the shape $w = sw's$, cf. Lemma 8.3. In what follows, we hope that this Lemma generalizes to arbitrary hypersquares.

The reader might expect how the strategy works in higher dimensions. Here we suppose that a similar conjecture as above holds true. Hence we have to determine the cohomology of $X(Q)$ for squares $Q \subset W$. So let Q be such a square with head $w = sw's$.

Case 1: Q is of the shape

$$\begin{array}{ccccc} & & w & & \\ & \nearrow & & \nwarrow & \\ Q: \quad sw' & & & & w's \ . \\ & \nwarrow & & \nearrow & \\ & & w' & & \end{array}$$

In this case the we have by Proposition 4.11 a splitting $H^i_c(X(Q)) = H^i_c(X(w's) \cup X(w')) \oplus H^{i-2}_c(X(w's) \cup X(w'))(-1)$. By induction the cohomology of $X(w's) \cup X(w')$ and hence of $X(Q)$ is known.

Case 2: Q is of the shape

$$\begin{array}{ccccc} & & w & & \\ & \nearrow & & \nwarrow & \\ Q: \quad v & & & & sw' \\ & \nwarrow & & \nearrow & \\ & & sv' & & \end{array}$$

with $v = sv's$. In this case the we have $H^i_c(X(Q)) = H^{i-2}_c(X(\hat{Q}))(-1)$ where $\hat{Q} = \{w's, v's, w', v'\}$. By induction the cohomology of $X(\hat{Q})$ and hence of $X(Q)$ is known.

Case 3: Q is of the shape

$$
Q: \quad \begin{tikzcd}[column sep=small, row sep=small] & w & \\ v_1 \arrow[ur] & & v_2 \arrow[ul] \\ & v_3 \arrow[ul] \arrow[ur] & \end{tikzcd}
$$

with $v_i = sv_i's$ for $i = 1, 2, 3$ and where $Q' := \{w', v_1', v_2', v_3'\} \subset W$ is a square. In this case the we consider as in the case of edges above the square $sQ' = \{sw', sv_1', sv_2', sv_0'\}$. Then the cohomology of $X(Q)$ sits in a long exact cohomology sequence

$$
\cdots \longrightarrow H_c^{i-1}(X(sQ') \longrightarrow H_c^i(X(Q)) \longrightarrow H_c^i(X(Q) \cup X(sQ')) \longrightarrow H_c^i(X(sQ')) \longrightarrow \cdots .
$$

Again by induction on the length of the head of a cube the cohomology of $X(sQ')$, $X(Q's)$ and $X(Q')$ are known. Further we have $H_c^i(X(Q) \cup X(sQ')) = H_c^{i-2}(X(Q's) \cup X(Q'))(-1)$, where $Q's := \{w's, v_1's, v_2's, v_3's\}$. Thus we may compute $r^i_{Q \cup sQ', sQ'}$ as in the lower dimensional cases, i.e. as in Conjecture 10.1 resp. Remark 10.2.

But there are yet two other kind of squares.

Case 4: Q is of the shape

$$
Q: \quad \begin{tikzcd}[column sep=small, row sep=small] & w & \\ sw' \arrow[ur] & & w's \arrow[ul] \\ & su' \arrow[ul] \arrow[ur] & \end{tikzcd}
$$

with $su' = u's$.

Case 5: Q is of the shape

$$
Q: \quad \begin{tikzcd}[column sep=small, row sep=small] & w & \\ sw' \arrow[ur] & & sv's \arrow[ul] \\ & su's \arrow[ul] \arrow[ur] & \end{tikzcd}
$$

with $u' < v' < w'$ and $\ell(w') = \ell(u') + 2$.

Unfortunately, as we see, the higher the dimension of a square is the more complicated the situation behaves. This is of course due to the relations which exist in the Weyl group

W. For this reason, we also consider the monoid F^+ in the sequel where this phenomena does not appear.

Before we proceed we recall the following well-known fact.

Lemma 10.4. *Let $w \in W$ and $s, t \in S$. Suppose that $\ell(sw) = \ell(w) + 1$, $\ell(wt) = \ell(w) + 1$ and $\ell(swt) = \ell(w)$. Then $w = swt$.*

Proof. This is [DL, Lemma 1.6.4] □

Thus if $\ell(sw') = \ell(w's) = \ell(w') + 1$ and $\ell(sw's) = \ell(w')$ we have $sw' = w's$.

Let $Q \subset W$ be a hypersquare with head $w = sw's$. There are a priori for the tail of Q the following possibilities (mod symmetry) where $u' \leq w'$.

Case A: $\mathrm{tail}(Q) = u'$ with $\ell(su's) = \ell(u') + 2$.

Case B: $\mathrm{tail}(Q) = u'$ with $\ell(su') = \ell(u's) = \ell(u') + 1$ and $\ell(su's) = \ell(u')$.

Case C: $\mathrm{tail}(Q) = su'$ with $\ell(su') = \ell(u') + 1$ and $\ell(su's) = \ell(u')$.

Case D: $\mathrm{tail}(Q) = su'$ with $\ell(su') = \ell(u') + 1$ and $\ell(su's) = \ell(u') + 2$.

Case E: $\mathrm{tail}(Q) = su's$ with $\ell(su's) = \ell(u') + 2$.

We shall examine in all cases the structure of Q and the cohomology of $X(Q)$.

Case A: $\mathrm{tail}(Q) = u'$ with $u' < w'$ and $\ell(su's) = \ell(u') + 2$.

Let $\dim(Q) = d$, so that $\#Q = 2^d$. We consider the subintervals $I(u, w), I(su', sw')$, $I(u's, w's), I(u', w')$. Since Q is a square each of them is a square as well and has consequently 2^{d-2} elements. We shall see that the union of them is Q. Indeed, let $v \in Q$.

Case 1) If $v \leq w'$ then $v \in I(u', w')$.

Case 2) Let $v \leq sw'$ and $v \not\leq w'$. Thus we may write $v = sv'$ with $v' \leq w'$. As $\ell(su') = \ell(u') + 1$ we see that by considering reduced decompositions that we must have $v \geq su'$. Thus $w \in I(su', sw')$.

Case 3) Let $v \leq w's$ and $v \not\leq w'$. This case is symmetric to Case 2, hence $w \in I(u's, w's)$.

Case 4) Let $v \leq sw's$ and $v \not\leq sw'$ and $v \not\leq w's$. Then as in Case 2 we argue that $v = sv's$ with $v' \geq u'$. Hence $v \in I(su's, sw's)$.

As $4 \cdot 2^{d-2} = 2^d$. we see that the pairwise intersection of the above 4 subsquares is empty and that $I(su', sw')$ (resp. $I(u, w), I(u's, w's)$) is induced by $I(u', w')$ by multiplying with s from the left (resp. conjugating with s, multiplying with s from the right). Hence Q is a

union of special squares (cf. Definition 6.4) and the cohomology is consequently given by

$$H^i_c(X(Q)) = H^i_c(X(Q(u', w's))) \oplus H^{i-2}_c(X(Q(u', sw')))(-1)$$

which is known by induction on the length.

Case B. $\mathrm{tail}(Q) = u'$ with $u' < w'$ and $\ell(su's) = \ell(u')$. It follows that $su' = u's$ by Lemma 10.4. We claim that this case does not appear. The case where $\dim(Q) = 2$ does not occur. Let $\dim(Q) = 3$. Then Q must have the shape

$$I(w, u') : \quad \begin{array}{ccc} & w & \\ sw' & sv's & w's \\ sv' & w' & v's \\ & u' & \end{array}$$

for some $v' \geq u'$ since $\ell(su') = \ell(u') + 1$ (consider a reduced decomposition of v'). Hence $v' = u'$. But then $sv's = su's$, a contradiction as $\ell(su's) = \ell(u')$.

If $\dim Q > 3$ then we argue by induction. Indeed in Q there must be a subsquare of dimension $\dim(Q) - 1$ with $\mathrm{head}(Q) = sv's$ and $\mathrm{tail}(Q) = u'$. By induction this is not possible.

Case C: $\mathrm{tail}(Q) = su'$ with $\ell(us') > \ell(u') < \ell(su')$.

We shall see that this case behaves very rigid. More precisely, we shall see that Q is paved by squares of type Case 4. Here we make usage of the following statement.

Lemma 10.5. *Let $w' \in W$, $s \in S$ with $\ell(sw's) = \ell(w') + 2$. Then there is no $v' \leq w'$ with $\ell(v') = \ell(w') - 1$ and such that $v' = su' = u's$ for some $u' \leq v'$.*

Proof. Let $w' = s_1 \cdots s_r$ be a reduced decomposition. Suppose that there exists such a $v' = su' = u's$ as above. Then there is some index $1 \leq i \leq r$ with $su' = s_1 \cdots \hat{s_i} \cdots s_r$. On the other hand, since $\ell(sv') < \ell(v')$ there exits by the *Exchange Lemma* some integer $1 \leq m \leq r$ with $ss_1 \cdots s_{m-1} = s_1 \cdots s_m$ (with s_i omitted depending on whether $i < m$ or $i > m$). If $m < i$, then $w' = s_1 \cdots s_r = ss_1 \cdots \hat{s_m} \cdots s_r$ a contradiction to the assumption that $\ell(sw') = \ell(w') + 1$. If $m > i$, then $ss_1 \cdots \hat{s_i} \cdots s_{m-1} = s_1 \cdots \hat{s_i} \cdots s_m$. But $su' = s \cdot s_1 \cdots \hat{s_i} \cdots \hat{s_m} \cdots s_r = s_1 \cdots \hat{s_i} \cdots \hat{s_m} \cdots s_r \cdot s$ as $su' = u's$. Hence we deduce that $s_{m+1} \cdots s_r \cdot s = s_m \cdot s_{m+1} \cdots s_r$. Again by plugging this expression into the reduced decomposition for w', we obtain a contradiction to the assumption that $\ell(w's) = \ell(w') + 1$. □

We start with the case of a square. Here it is as in Case 4 before. Consider now a hypersquare Q of dimension 3. Thus it must have the shape

$$Q:\quad \begin{array}{ccc} & w & \\ sw' & sv's & w's \\ ? & ? & ? \\ & su' & \end{array}$$

with $su' = u's$, $v' < w'$ and $u' < w'$ for certain elements $? \in W$. As $\ell(su') = \ell(u') + 1$ we deduce that $v' \geq u'$. Hence we can make the structure of Q more precise, i.e.

$$Q:\quad \begin{array}{ccc} & w & \\ sw' & sv's & w's \\ sv' & ? & v's \\ & su' & \end{array}$$

As one verifies there are a priori for $? \in W$ only 2 possibilities : $? \in \{w', sz'\}$ with $sz' = z's$ and $z' \in I(u', w')$. But by Lemma 10.5 we must have $? = sz'$. Hence Q is the union of two squares of the shape as in Case 4.

Let $\dim(Q) = d > 3$. The square Q begins with the following elements

$$Q:\quad \begin{array}{ccc} & w & \\ sw' & sv'_1 s, \cdots, sv'_{d-2}s & w's \\ \vdots & \vdots & \vdots \\ & su' & \end{array}$$

By induction on the size of Q we know that all subsquares of Q of the shape $Q(su', sv'_i s)$ are union of squares of the the desired shape. Now one verifies that square Q ends as follows

$$Q:\quad \begin{array}{ccc} & w & \\ \vdots & \vdots & \vdots \\ sx' & sy'_1, \cdots, sy'_{d-2} & x's \\ & su' & \end{array}$$

with $sy_i = y_i s,\ i = 1, \ldots, d$. There must be some y_i with $sy' \leq w$ and again by induction the statement follows.

Next we turn to the cohomology with respect to these hypersquares. We start again with the 2-dimensional case. We fix reduced decompositions of w' and u'. We consider the hypersquare $Q^{F^+}(u', w) \subset F^+$

$$Q^{F^+}(u',w): \quad \begin{tikzcd} & & w & & \\ sw' \arrow[urr] & & su's \arrow[u] & & sw' \arrow[ull] \\ su' \arrow[u] \arrow[urr] & & w' \arrow[ull] \arrow[urr] & & u's \arrow[u] \arrow[ull] \\ & & u' \arrow[ull] \arrow[u] \arrow[urr] & & \end{tikzcd}$$

and the interval $I(u',w)$ in W. A case by case study together with the fact (which follows by Lemma 10.4) that apart from u' there is no $z' \prec w'$ with $\ell(z') = \ell(w') - 1$ and $\gamma(z') = \gamma(u')$ it is seen that the preimage of $I(u',w)$ under the proper map $\pi : X^{\ell(w)+1} \longrightarrow X$ is just $X(Q^{F^+}(u',w))$. Hence we get a proper surjective map

$$\pi : X(Q^{F^+}(u',w)) \longrightarrow X(I(u',w)).$$

The hypersquare Q is an open subset of the interval $I(u',w)$ so that $U := X(Q)$ is an open subvariety of $X(I(u',w))$. The closed complement is given by $Y := X(I(u',w)) \setminus X(Q)$. We consider their preimages $U' := \pi^{-1}(X(Q))$ and $Y' := \pi^{-1}(Y)$ in $X^{\ell(w)+1}$. We get a commutative diagram of long exact cohomology sequences

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^i_c(U') & \longrightarrow & H^i_c(X(Q^{F^+}(u',w))) & \longrightarrow & H^i_c(Y') & \longrightarrow \cdots \\ & \uparrow & & \uparrow & & \uparrow & \\ \cdots \longrightarrow & H^i_c(X(Q)) & \longrightarrow & H^i_c(X(I(u',w)) & \longrightarrow & H^i_c(Y) & \longrightarrow \cdots \end{array} \tag{10.1}$$

We claim that we can recover the cohomology of $X(Q)$ by this diagram. In fact, the cohomology of $X(Q^{F^+}(u',w))$ is known by its particular structure since it is the union of special squares, i.e.

$$H^i_c(X(Q^{F^+}(u',w))) = H^i_c(X(Q^{F^+}(u',w's))) \oplus H^{i-2}_c(X(Q^{F^+}(u',w's)))(-1).$$

Further $Y = X(w') \cup X(u')$ whereas $Y' = X_1^{F^+}(su's) \cup Y^{F^+}$ with the obvious meaning for Y^{F^+} and $X_1^{F^+}(su's)$ is the closed subset of $X^{F^+}(su's)$ in Remark 4.14. Then

$$H^i_c(Y') = H^i_c(Y) \oplus H^{i-2}_c(X(u'))(-1)$$

and

$$H^i_c(U') = H^i_c(U) \oplus H^{i-2}_c(X(u's))(-1).$$

Hence the restriction map $H^i_c(X(Q^{F^+}(u',w))) \longrightarrow H^i_c(Y')$ is induced by the sum of the maps

$$H^i_c(X(Q(u',w's))) \longrightarrow H^i_c(Y) \text{ and } H^{i-2}_c(X(Q(u',w's)))(-1) \longrightarrow H^{i-2}_c(X(u'))(-1).$$

The latter one factories over the representation $H_c^{i-2}(X(w') \cup X(v'))(-1)$. Hence both maps are known by induction. Thus we deduce the cohomology of U'. By factoring out the second summand in $H_c^i(U') = H_c^i(U) \oplus H_c^{i-2}(X(u's))(-1)$ we get the cohomology of $X(Q)$. Furthermore, the boundary map $H_c^{i-2}(X(u')) \longrightarrow H_c^{i-1}(X(u's))$ which appears in the boundary map $H_c^i(Y') \longrightarrow H_c^{i+1}(U')$ is known, as well.

If $\dim(Q) > 2$ then one verifies that the above description generalizes to the higher dimensional setting. In particular, we get

$$H_c^i(Y') = H_c^i(Y) \oplus H_c^{i-2}(\bigcup_{\substack{sv' \in Q \\ sv' = v's}} X(v'))(-1)$$

and

$$H_c^i(U') = H_c^i(U) \oplus H_c^{i-2}(\bigcup_{\substack{sv' \in Q \\ sv' = v's}} X(sv'))(-1).$$

The second summand is known by induction as the set $\{sv' \in Q \mid sv' = v's\}$ forms a subsquare in W.

For the remaining two cases, we introduce the following partial order on the sets of squares of type Case 1 - 5 via the following pre-order diagram.

$$\begin{array}{ccccc} & & \text{Case 3} & & \\ & & \downarrow & & \\ & & \text{Case 5} & & \\ & \swarrow & & \searrow & \\ \text{Case 4} & & & & \text{Case 2} \\ & & & & \downarrow \\ & & & & \text{Case 1} \end{array}$$

Here the arrow Case i $\longrightarrow$ Case j means that Case j $<$ Case i.

Case D: $\mathrm{tail}(Q) = su'$ with $u' < w'$ and $\ell(su's) = \ell(u') + 2$.

We start with the case of a square. Here it has the shape as in Case 2 before. Consider now a hypersquare Q of dimension 3. Thus it must have the shape

$$Q: \quad \begin{array}{ccc} & w & \\ sw' & sv's & ? \\ sv' & ? & su's \\ & su' & \end{array}$$

for certain elements $? \in W$ where $v' \geq u'$ as $\ell(u's) = \ell(u') + 1$. There are two possibilities for completing Q. If $w' \geq su'$, then we get

$$Q: \quad \begin{array}{ccc} & w & \\ sw' & sv's & w's \\ sv' & w' & su's \\ & su' & \end{array}.$$

On the other hand, if $w' \not\geq su'$, then we get

$$Q: \quad \begin{array}{ccc} & w & \\ sw' & sv's & sz's \\ sv' & sz' & su's \\ & su' & \end{array}.$$

Hence if we write $Q = Q(su', sv's) \dot\cup Q'$ then Q' is a specialization of $Q(su', sv's)$, i.e. $Q' \leq Q(su', sv's)$. For $\dim(Q) = d > 3$, we claim that Q is paved by 3-dimensional hypersquares of this kind. More precisely, if $Q' \subset Q$ is a 3-dimensional subsquare which we write as $Q = Q_1 \dot\cup Q_2$ where $\mathrm{head}(Q_i) = sv_is$ with $v_2 \prec v_1$, then $Q_1 \leq Q_2$. Indeed, the square Q begins with the following elements

$$Q: \quad \begin{array}{ccc} & w & \\ sw' & sv'_1s, \cdots, sv'_{d-2}s & ? \\ \vdots & \vdots & \vdots \\ & su' & \end{array}$$

with $? \in \{w's, sv'_{d-1}s\}$. By induction on the size of Q we know that all subsquares of Q of the shape $Q(su', sv'_is)$ are union of squares of the the desired shape. But the union over all these squares exhaust all apart from $\{w, sw', ?, ?\}$. Indeed the number of elements in this union is $2^{d-1} + 2^{d-2} + \cdots + 2^2 = 2^d - 4$. Again if $w' \geq su'$, then $\{w, sw', ?, ?\} = Q_w$ and the claim follows. On the other hand if $w' \not\geq su'$, then we can even says that Q is paved by squares of type Case 2, since there cannot be an element $v' \leq w'$ with $v' \in Q$ and $v' \geq su'$.

Let's determine the cohomology of $X(Q)$. If $\dim(Q) = 2$, then we get

$$H^i_c(Q(su', w)) = H^{i-2}_c(Q(u', w's))(-1).$$

If $\dim(Q) = 3$ and $w' \not\geq su'$ then again - by the observation above - we get $H^i_c(Q(su', w)) = H^{i-2}_c(Q(u', w's))(-1)$. Consider the other possibility of a cube. Here we consider as in the previous case the hypersquare $\hat{Q} := Q^{F^+}(u', w)$ in F^+ and the interval $I(u', w)$ in W.

$$\hat{Q}: \quad \begin{array}{cccccc} & & & w & & \\ & sw' & sv's & w's & sz's & \\ sv' & w' & su's & v's & sz' & z's \\ & su' & v' & u's & z' & \\ & & & u' & & \end{array}$$

with $z' = su'$. The map $\pi : X^{\ell(w)+1} \longrightarrow X$ induces surjective map

$$\pi : X(Q^{F^+}(u',w)) \longrightarrow X(I(u',w))$$

which is even proper although $X(Q^{F^+}(u',w))$ might be strictly contained in the subset $\pi^{-1}(X(I(u',w)))$. (The reason is that for any closed subset $A \subset \pi^{-1}(X(I(u',w)))$ the identity $\pi(A) = \pi(A \cap X(Q^{F^+}(u',w)))$ holds). We set $U := X(Q) \subset X(I(u',w))$ and $Y := X(I(u',w)) \setminus X(Q)$. We consider their preimages $U' := \pi^{-1}(X(Q))$ and $Y' := \pi^{-1}(Y)$ in $X(Q^{F^+}(u',w))$. Again we claim that we can recover the cohomology of $X(Q)$ by the diagram 10.1. The reasoning is similar to Case C. In fact, we have

$$H^i_c(X(Q^{F^+}(u',w))) = H^i_c(X(Q^{F^+}(u',w's))) \oplus H^{i-2}_c(X(Q^{F^+}(u',w's)))(-1)$$

since $Q^{F^+}(u',w)$ is paved by special squares. Further

$$Y = X(u') \cup X(v') \cup X(u's) \cup X(v's)$$

and

$$Y' = Y^{F^+} \cup X_1^{F^+}(sz') \cup X^{F^+}(sz's)$$

whereas

$$U' = U^{F^+} \cup X^{F^+}(z') \cup X^{F^+}(z's) \cup X_2^{F^+}(sz') \cup X_2^{F^+}(sz's).$$

Here $X_2^{F^+}(su's)$ is the open subset of $X^{F^+}(su's)$ in Remark 4.14. Then

$$H^i_c(Y') = H^i(Y) \oplus H^{i-2}_c(X(u') \cup X(u's))(-1)$$

and

$$H^i_c(U') = H^i_c(U) \oplus H^{i-2}_c(X(su's) \cup X(su'))(-1).$$

The restriction map $H^i_c(X(Q^{F^+}(u',w))) \longrightarrow H^i_c(Y')$ is given by the sum of the maps

$$H^i_c(X(Q(u',w's))) \longrightarrow H^i_c(Y) \text{ and } H^{i-2}_c(X(Q(u',w's)))(-1) \longrightarrow H^{i-2}_c(X(u's) \cup X(u'))(-1).$$

Again both maps are known by induction. Thus we deduce the cohomology of U'. By factoring out the second summand in $H^i_c(U') = H^i_c(U) \oplus H^{i-2}_c(X(su's) \cup X(su'))(-1)$ we get the cohomology of $X(Q)$.

The higher dimensional case is treated similar as in Case C.

Case E: $\mathrm{tail}(Q) = su's$ with $u' < w'$ and $\ell(su's) = \ell(u') + 2$.

We start with the case of a square. Here it has the shape as in Case 3 or Case 5 before. Consider now a hypersquare Q of dimension 3. Thus if its lower subsquare is as in Case 5 must have the shape

$$Q: \quad \begin{array}{ccc} & w & \\ ? & sv's & ? \\ sv' & ? & sz's \\ & su's & \end{array}$$

for certain elements $? \in W$. There are three possibilities for completing Q. If $w' \geq su'$, then we get

$$Q: \quad \begin{array}{ccc} & w & \\ sw' & sv's & w's \\ sv' & sy' & sz's \\ & su's & \end{array}$$

with $sy' = y's$. On the other hand, if $w' \not\geq su'$, then we get

$$Q: \quad \begin{array}{ccc} & w & \\ sw' & sv's & sy's \\ sv' & ? & sz's \\ & su's & \end{array}$$

with $? = sy'$ or $? = sx's$ for some $x' < v'$. Hence if we write $Q = Q(su's, sv's) \dot\cup Q'$ then Q' is a specialization of $Q(su's, sv's)$, i.e. $Q' \leq Q(su', sv's)$. For $\dim(Q) = d > 3$, one proves as in Case D that is paved by 3-dimensional hypersquares of this kind.

Consider now a hypersquare Q of dimension 3 such that its lower subsquare is as in Case 3,

$$Q: \quad \begin{array}{ccc} & w & \\ ? & sv's & ? \\ sx's & ? & sy's \\ & su's & \end{array}$$

for certain elements $? \in W$. This is the most generic case in the sense that the upper square in Q can be arbitrary, i.e. if we write $Q = Q(su's, sv's) \dot\cup Q'$ then Q' is a any specialization of $Q(su', sv's)$.

Now we consider the cohomology and discuss only the case of a square Q. The higher dimensional cases are treated as before, see also Remark 10.6 for a general approach. So let Q be a square as in Case 5 (Case 3 has been already explained)

$$Q: \quad \begin{tikzcd} & w & \\ sw' \arrow[ur] & & sv's \arrow[ul] \\ & su's \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where $$\begin{tikzcd} & w' & \\ y' \arrow[ur] & & v' \arrow[ul] \\ & u' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

is a square with $y' = u's$. We consider the extended interval $I(su', w) \subset W$ resp. the cube $\hat{Q} := Q^{F^+}(su', w) \subset F^+$.

$$\hat{Q}: \quad \begin{tikzcd} & w & \\ sw' \arrow[ur] & sv's \arrow[u] & sy's \arrow[ul] \\ sv' \arrow[u] \arrow[ur] & su's \arrow[ul] \arrow[ur] & sy' \arrow[u] \arrow[ul] \\ & su' \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd} .$$

Here

$$Y = X(su') \cup X(sv').$$

$$Y' = Y^{F^+} \cup X_1^{F^+}(sy's).$$

$$U = X(w) \cup X(sw') \cup X(sv's) \cup X(su's).$$

$$U' = U^{F^+} \cup X^{F^+}(sy') \cup X_2^{F^+}(sy's).$$

Hence we get

$$H_c^i(Y') = H_c^i(Y) \oplus H_c^{i-2}(X(u's))(-1)$$

and

$$H_c^i(U') = H_c^i(U) \oplus H_c^{i-2}(X(su's))(-1).$$

The restriction map $H^i_c(X(Q^{F^+}(su',w))) \longrightarrow H^i_c(Y')$ is given as follows. First note that $H^i_c(X(\hat{Q})) = H^{i-2}_c(X(s\backslash Q))(-1)$, where $s\backslash Q$ is the cube such that $s\cdot(s\backslash Q) = Q$. With respect to the summand $H^i_c(Y)$ we know the map factorizes over $H^i_c(X(Q^{F^+}(su',sw'))) \longrightarrow H^i_c(X(sv')\cup X(su'))$. As for the summand $H^{i-2}_c(X(su'))(-1)$ the necessary information follows from the identity $H^i_c(X(\hat{Q})) = H^{i-2}_c(X(s\backslash Q))(-1)$. All maps are known. On the other hand we know by induction the boundary map $H^{i-2}_c(X(su')) \longrightarrow H^{i-1}_c(X(su's))$. Again we deduce the cohomology of U' and by factoring out the summand $H^{i-2}_c(X(su's))(-1)$ we get the cohomology of U.

Thus we have examined all cases. In remains to say that the start of induction is the situation where head(Q) is minimal in its conjugacy class. This case can be handled explicitly using successive Proposition 5.8.

Remark 10.6. Let $I = I(u,w) \subset W$ be any interval. The map $\pi : X^{\ell(w)+1} \longrightarrow X$ induces a proper map

$$\pi : \pi^{-1}(X(I)) \longrightarrow X(I).$$

The following lines gives a description of the preimage $Z \subset X^{\ell(w)+1}$. Let $v \in Q^{F^+}(1,w)$.

1. Case. $\ell(\gamma(v)) = \ell(v)$.

Subcase a) $\gamma(v) \not\geq u$. In this case $X^{F^+}(v)\cap Z = \emptyset$.

Subcase b) $\gamma(v) \geq u$. In this case $X^{F^+}(v) \subset Z$ and the restriction of π to $X^{F^+}(v)$ induces an isomorphism $X^{F^+}(v) \xrightarrow{\sim} X(\gamma(v))$.

2. Case. $\ell(\gamma(v)) < \ell(v)$. By Lemma 7.1 we may suppose that $v = v_1\cdot t\cdot t\cdot v_2$. Thus we may write $X^{F^+}(v) = X^{F^+}_1(v)\bigcup X^{F^+}_2(v)$ where $X^{F^+}_1(v)$ is closed and $X^{F^+}_2(v)$ is open. We have $\mathbb{A}^1$-bundles $X^{F^+}_1(v) \longrightarrow X^{F^+}(v_1v_2)$ and $X^{F^+}_2(v)\cup X^{F^+}(v_1tv_2) \longrightarrow X^{F^+}(v_1tv_2)$. The map $\pi|_{X^{F^+}} : X^{F^+}(v) \longrightarrow X$ factorizes through $X^{F^+}(v_1tv_2)\cup X^{F^+}(v_1v_2)$. Hence we have reduced the question to elements of lower length.

Suppose additionally that $w = sw's$. Then $Q^{F^+}(1,w)$ is paved by special squares. Then it is possible to say what the image of such a special square $Q_v = \{v, sv', v's, v'\}$ under the map π is. But we do not carry out this since there are to many cases.

11. Appendix B

Tables of the cohomology of Weyl group elements of full support in GL_3 and GL_4. Here we list only representatives of cyclic shift classes.

GL_3

w	$H_c^*(X(w))$
$(1,2,3)$	$j_{(1,1,1)}[-2] \oplus j_{(2,1)}(-1)[-3] \oplus j_{(3)}(-2)[-4]$
$(1,3)$	$j_{(1,1,1)}[-3] \oplus j_{(3)}(-3)[-6]$

GL_4

w	$H_c^*(X(w))$
$(1,2,3,4)$	$j_{(1,1,1,1)}[-3] \oplus j_{(2,1,1)}(-1)[-4] \oplus j_{(3,1)}(-2)[-5] \oplus j_{(4)}(-3)[-6]$
$(1,2,4)$	$j_{(1,1,1,1)}[-4] \oplus j_{(2,2)}(-2)[-5] \oplus j_{(4)}(-4)[-8]$
$(1,3)(2,4)$	$j_{(1,1,1,1)}[-4] \oplus j_{(2,2)}(-1)[-4] \oplus j_{(2,1,1)}(-2)[-5] \oplus j_{(3,1)}(-2)[-5] \oplus$ $j_{(2,2)}(-3)[-6] \oplus j_{(4)}(-4)[-8]$
$(1,3,2,4)$	$j_{(1,1,1,1)}[-5] \oplus j_{(2,2)}(-2)[-6] \oplus j_{(2,1,1)}(-2)[-6] \oplus j_{(2,2)}(-3)[-7] \oplus$ $j_{(3,1)}(-3)[-7] \oplus j_{(4)}(-5)[-10]$
$(1,4)$	$j_{(1,1,1,1)}[-5] \oplus j_{(2,1,1)}(-1)[-5] \oplus j_{(2,2)}(-2)[-6] \oplus j_{(2,2)}(-3)[-7] \oplus$ $j_{(3,1)}(-4)[-8] \oplus j_{(4)}(-5)[-10]$
$(1,4)(2,3)$	$j_{(1,1,1,1)}[-6] \oplus j_{(2,1,1)}(-2)[-7] \oplus j_{(2,2)}(-3)[-8]^2 \oplus j_{(3,1)}(-4)[-9] \oplus j_{(4)}(-6)[-12].$

References

[BGG] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, *Differential operators on the base affine space and a study of g-modules*. Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), pp. 21 - 64, Halsted, New York, 1975.

[BL] S. Billey, V. Lakshmibai, *Singular loci of Schubert varieties. Progress in Mathematics*, **182**. Birkhäuser Boston, Inc., Boston, MA, 2000.

[Bl] S. Bloch, *Algebraic cycles and higher K-theory*. Adv. in Math. **61** (1986), no. 3, 267–304.

[BM] M. Broué, J. Michel, *Sur certains éléments réguliers des groupes de Weyl et les variéts de Deligne-Lusztig associées*, Finite reductive groups (Luminy, 1994), 73–139, Progr. Math., **141**, Birkhäuser Boston, Boston, MA, 1997.

[De] P. Deligne, *Action du groupe des tresses sur une catgorie*, Invent. Math. **128** (1997), no. 1, 159–175.

[Du] O. Dudas, *Cohomology of Deligne-Lusztig varieties for short-length regular elements in exceptional groups*, preprint 2011.

[DL] P. Deligne, G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.

[DM] F. Digne, J. Michel, *Endomorphisms of Deligne-Lusztig varieties*, Nagoya Math. Journal **183** (2006), 35–103.

[DMR] F. Digne, J. Michel, R. Rouquier, *Cohomologie des variétés de Deligne-Lusztig*, Adv. Math. **209** (2007), No. 2, 749-822.

[DOR] J.-F. Dat, S. Orlik, M. Rapoport, *Period domains over finite and p-adic fields*, Cambridge Tracts in Mathematics (No. **183**), Cambridge University Press, 2010.

[Dr] V.G. Drinfeld, *Coverings of p-adic symmetric domains*, Funkcional. Anal. i Priloen. **10** (1976), no. 2, 29–40.

[FH] W. Fulton, J. Harris, *Representation theory. A first course*. Graduate Texts in Mathematics, **129** Readings in Mathematics. Springer-Verlag, New York, 1991.

[Fu] W. Fulton, *Intersection theory*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 2. Springer-Verlag, Berlin, 1998.

[Ge] A. Genestier, *Espaces symétriques de Drinfeld*, Astérisque No. **234** (1996).

[GK] E. Grosse-Klönne, *Integral structures in the p-adic holomorphic discrete series*. Represent. Theory **9** (2005), 354–384.

[GP] M. Geck, G. Pfeiffer, *On the irreducible characters of Hecke algebras*, Adv. Math. **102** (1993), no. 1, 79–94.

[GP'] M. Geck, G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, London Mathematical Society Monographs. New Series, **21**. The Clarendon Press, Oxford University Press, New York, 2000.

[GKP] M. Geck, S. Kim, G. Pfeiffer, *Minimal length elements in twisted conjugacy classes of finite Coxeter groups*, J. Algebra **229** (2000), no. 2, 570–600.

[Ho] R. Howe, *Harish-Chandra homomorphisms for p-adic groups*. With the collaboration of Allen Moy. CBMS Regional Conference Series in Mathematics, 59. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1985.

[Hu] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics, E30. Friedr. Vieweg und Sohn, Braunschweig, 1996.

[I] T. Ito, *Weight-monodromy conjecture for p-adically uniformized varieties*, Invent. Math. **159** (2005), no. 3, 607–656.

[Ku] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*. Progress in Mathematics, **204**. Birkhuser Boston, Inc., Boston, MA, 2002.

[Le] G. I. Lehrer, *The spherical building and regular semisimple elements*, Bull. Austral. Math. Soc. **27** (1983), no. 3, 361379.

[L] G. Lusztig, *Representations of finite Chevalley groups*, Expository lectures from the CBMS Regional Conference held at Madison, Wis., August 8–12, 1977. CBMS Regional Conference Series in Mathematics, **39**. American Mathematical Society, Providence, R.I., 1978.

[L2] G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius*, Invent. Math. **38** (1976/77), no. 2, 101–159.

[L3] G. Lusztig, *Characters of reductive groups over a finite field*. Annals of Mathematics Studies, 107. Princeton University Press, Princeton, NJ, 1984.

[M] J. Milne, *Lectures on etale cohomology*, http://www.jmilne.org/math/.

[O] S. Orlik, *Kohomologie von Periodenbereichen über endlichen Körpern*, J. Reine Angew. Math. **528** (2000), 201–233.

[SGA5] Séminaire de Géomtrie Algébrique du Bois Marie - 1965-66 - Cohomologie l-adique et Fonctions L - (SGA 5). Lecture notes in mathematics. 589. Berlin; New York: Springer-Verlag.

[SS] P. Schneider, U. Stuhler, *The cohomology of p-adic symmetric spaces*, Invent. Math. **105** (1991), 47–122.

Bergische Universität Wuppertal, Fachbereich C - Mathematik und Naturwissenschaften, Gaussstr. 20, 42119 Wuppertal, Germany.

E-mail address: orlik@math.uni-wuppertal.de